Weighted norm inequalities for a general maximal operator

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1. Introduction

Some studies about boundedness properties of maximal operators of Hardy—Littlewood type have been made recently (see [5], [6], [8]). In this note we study a very general operator which includes the known results as particular cases.

Let U, V be two arbitrary sets. Suppose we have some topological structure in the cartesian products $\mathbf{R}^n \times U$, $\mathbf{R}^n \times V$ and also suppose the existence of positive Borel measures $d\alpha(x, u)$ on $\mathbf{R}^n \times U$ and $d\beta(x, v)$ on $\mathbf{R}^n \times V$.

We shall denote by $L^p(\mathbf{R}^n \times U, d\alpha)$ the set of measurable functions in $\mathbf{R}^n \times U$ such that $\int_{\mathbf{R}^n \times U} |f(x, u)|^p \, d\alpha(x, u)$ is finite.

The α-measure of a set $E \subset \mathbf{R}^n \times U$ will be indicated by $\alpha(E)$ and the Lebesgue measure of $E \subset \mathbf{R}^n$ will be denoted by $|E|$.

Throughout this paper Φ, respectively Ψ, will be a set function from cubes in $\mathbf{R}^n$ into Borel sets in $\mathbf{R}^n \times U$, resp. $\mathbf{R}^n \times V$, satisfying:

(I) If Q_1, Q_2 are cubes with $Q_1 \cap Q_2 = \emptyset$ then $\Phi(Q_1) \cap \Phi(Q_2) = \emptyset$ and $\Psi(Q_1) \cap \Psi(Q_2) = \emptyset$.

(II) If $Q_1 \subset Q_2$ then $\Phi(Q_1) \subset \Phi(Q_2)$ and $\Psi(Q_1) \subset \Psi(Q_2)$.

(III) If $Q(x, r)$ denotes the cube with center x and side length r then, for any $x \in \mathbf{R}^n$

$$\bigcup_{r>0} \Phi(Q(x, r)) = \mathbf{R}^n \times U \quad \text{and} \quad \bigcup_{r>0} \Psi(Q(x, r)) = \mathbf{R}^n \times V.$$

We define the following maximal operator which applies functions in $\mathbf{R}^n \times U$ into functions in $\mathbf{R}^n \times V$:

$$Tf(x, v) = \sup \left\{ \frac{1}{|Q|} \int_{\Phi(Q)} |f(y, u)| \, d\alpha(y, u) \colon (x, v) \in \Psi(Q) \right\} \tag{1}$$

i.e. the supremum is taken over all cubes Q such that $(x, v) \in \Psi(Q)$.