Analytic approximability of solutions of partial differential equations

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1. Introduction

Let $P(x, D_x)$ be a linear partial differential operator with analytic coefficients defined in a neighborhood of a point $x_0 \in \mathbf{R}^n$. We shall call P *locally approximable* at x_0 if for any distribution u for which $Pu \equiv 0$ in a neighborhood of x_0, there is a neighborhood $\mathcal{U}$ of x_0 and a sequence of distributions u_j real analytic in $\mathcal{U}$ such that

$$u_j \to u \quad \text{in} \quad \mathcal{U},$$
$$Pu_j \equiv 0 \quad \text{in} \quad \mathcal{U}.$$

The property of local approximability was studied by Baouendi and Treves [2], who showed that P is locally approximable at x_0 if its complex characteristics at x_0 are simple. Métivier [7] has proved approximability for a class of first order nonlinear equations. Baouendi and the second author [1] showed that any left invariant differential operator on a Lie group is locally approximable.

The class of locally approximable differential operators contains that of analytic hypoelliptic differential operators. (Recall that P is analytic hypoelliptic at x_0 if Pu real analytic in a neighborhood of x_0 implies that u is real analytic near x_0.) The notion of analytic hypoellipticity has been microlocalized in an obvious way, but the notion of microlocal approximability is less clear. In § 2 we give a definition of microlocal approximability and also extend the definition of local approximability to pseudodifferential operators. These definitions are based on the constants for the Fourier—Bros—Iagolnitzer transform of a distribution (see e.g. [11]). We show that when $\operatorname{char}_{x_0} P$ is contained in a line then local approximability is equivalent to microlocal approximability in all directions.

* Partially supported by N.S.F. grant DMS 8601260.