Whitney's extension theorem for ultradifferentiable functions of Beurling type

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Classes of non-quasianalytic functions on $\mathbf{R}^n$ are usually defined by imposing conditions on the derivatives of the functions. For example, if $(M_p)_{p\in\mathbf{N}_0}$ is an appropriate sequence of positive numbers, one defines

$$\mathscr{E}^{(M_p)}(\mathbf{R}^n) := \{f\in C^\infty(\mathbf{R}^n) \mid \text{ for each compact set } K \text{ in } \mathbf{R}^n \text{ and each } h>0 \quad \sup_{\alpha\in\mathbf{N}_0^n}\sup_{x\in K}|f^{(\alpha)}(x)|(h^{|\alpha|}M_{|\alpha|})^{-1}<\infty\};$$

$\mathscr{E}^{\{M_p\}}(\mathbf{R}^n)$ is defined similarly (replacing the all quantifier over h by an existence quantifier). Continuing the classical work of E. Borel [5] many authors (see Bronshtein [7], Bruna [8], Carleson [9], Dzanasija [10], Ehrenpreis [11], Komatsu [15], Mityagin [22], Petzsche [24] and Wahde [30]) have investigated conditions on $(M_p)_{p\in\mathbf{N}_0}$ and on sequences $(a_\alpha)_{\alpha\in\mathbf{N}_0^n}$ implying the existence of $f\in\mathscr{E}^{(M_p)}(\mathbf{R}^n)$ (resp. $\mathscr{E}^{\{M_p\}}(\mathbf{R}^n)$) with

$$f^{(\alpha)}(0) = a_\alpha \quad \text{for all} \quad \alpha\in\mathbf{N}_0^n.$$

In the present note we study this question and a version of Whitney's extension theorem for the non-quasianalytic classes $\mathscr{E}_\omega(\mathbf{R}^n)$ which have been introduced by Beurling [2] and Björck [3] using the Fourier transform. Most familiar function classes, like the Gevrey classes, can be obtained by both methods ($M_p=(p!)^s$ or $\omega(x)=|x|^{1/s}$, $s>1$). However, in general, the two definitions lead to different classes.

To define $\mathscr{E}_\omega(\mathbf{R}^n)$ we vary Beurling's approach a bit. We assume that $\omega\colon \mathbf{R}\to[0,\infty[$ is a continuous function having the following properties:

(i) $$\limsup_{t\to\infty}\frac{\omega(2t)}{\omega(t)}<\infty \quad\text{and}\quad \lim_{t\to\infty}\frac{\log t}{\omega(t)}=0;$$

(ii) $$\int_{-\infty}^{+\infty}\frac{\omega(t)}{1+t^2}\,dt<\infty;$$

(iii) $\varphi\colon t\mapsto\omega(e^t)$ is a convex function on $\mathbf{R}$.