A-superharmonic functions and supersolutions of degenerate elliptic equations

Juha Heinonen and Tero Kilpeläinen

1. Introduction

We consider supersolutions of the equation

$$\nabla \cdot A(x, \nabla u) = 0 \tag{1.1}$$

where $A: G \times \mathbf{R}^n \to \mathbf{R}^n$ is a strictly monotone (usually non-linear) elliptic differential operator in an open set G in $\mathbf{R}^n$, $n \geqq 2$. The precise assumptions are given in Section 2. In connection with equations of the type (1.1) we refer, for example, to [9], [11], [15], [20], and [21]. Supersolutions of (1.1) are the functions $u \in \operatorname{loc} W_p^1(G)$ satisfying

$$\int_G A(x, \nabla u) \cdot \nabla \varphi \, dx \geqq 0$$

for all non-negative $\varphi \in C_0^\infty(G)$. Supersolutions in general fail to be continuous and, in order to have pointwise estimates, the above definition is not quite adequate. It is our purpose in this paper to show that the classical potential theoretical definition for superharmonic functions is pertinent also in non-linear situations, and that it indeed yields a class of functions which strictly includes the supersolutions of (1.1) and is closed under upper directed monotone convergence. More precisely, we say that a lower semicontinuous function $u: G \to \mathbf{R} \cup \{\infty\}$ is *A-superharmonic* if it satisfies the comparison principle: for each domain $D \subset\subset G$ and each function $h \in C(\bar{D})$ which is a solution of (1.1) in D, the condition $h \leqq u$ in ∂D implies $h \leqq u$ in D. The comparison principle is valid for solutions of (1.1), whence potential theoretical aspects can be salvaged.

It is shown that supersolutions of (1.1) can be redefined in a set of measure zero so that they become A-superharmonic and, conversely, that if u is a locally bounded A-superharmonic function, then u belongs locally to the Sobolev space W_p^1 and is a supersolution of (1.1). To sum up, we may say that A-superharmonic functions form a closure of supersolutions with respect to upper directed monotone