Mean oscillation and commutators of singular integral operators

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0. Introduction

Let T be a Caldéron—Zygmund transform

$$Tg(x) = \text{P.V.} \int_{R^d} K(x-y)\, g(y)\, dy$$

where the kernel K is homogeneous of degree $-d$, i.e. $K(x) = |x|^{-d} K(x/|x|)$, $\int_{S^{d-1}} K = 0$ and K satisfies some smoothness condition. $K \in C^\infty(S^{d-1})$ will always be sufficient. For the theory of these transforms, see e.g. Stein [7]. We need the result that T is bounded on L^p, $1<p<\infty$. K and T will be fixed throughout the paper and not identically zero.

Let f be a function on R^d, and let it also denote the operation of pointwise multiplication with f. We will study the commutator $[f, T]$ denoted by C_f.

Formally

$$\begin{aligned} C_f g(x) &= fTg(x) - Tfg(x) \\ &= f(x) \int K(x-y)\, g(y)\, dy - \int K(x-y) f(y)\, g(y)\, dy \\ &= \int (f(x) - f(y)) K(x-y)\, g(y)\, dy. \end{aligned}$$

For these formulas to make sense, f has to be locally integrable. $C_f g$ is then defined a.e. as a principal value for g bounded and with compact support. C_f may be extended to all of L^p when we have proved it to be continuous. $C_f g$ is clearly bilinear.

Let Q be any cube in R^d. We define f_Q, the mean value of f on Q, as

$$|Q|^{-1} \int_Q f(x)\, dx$$

and $\Omega(f, Q)$, the mean oscillation of f on Q, as

$$|Q|^{-1} \int_Q |f - f_Q|\, dx.$$