Local surjectivity in C^∞ for a class of pseudo-differential operators

Paul Godin*

Let X be a C^∞ paracompact n-dimensional manifold and P a properly supported pseudo-differential operator on X, having a principal symbol p homogeneous of degree m. P is said to be locally solvable at $x_0 \in X$ if there is a neighbourhood V of x_0 such that for all $f \in C^\infty(X)$, there exists $u \in \mathcal{D}'(X)$ satisfying $Pu=f$ in V. P is said to be of principal type if for all $(x, \xi) \in T^*X \setminus 0$, one has $\partial p/\partial \xi \neq 0$. For operators of principal type, Beals and Fefferman [1] proved that the so-called condition $(\mathscr{P})$ is sufficient for local solvability. On the other hand, Egorov [1] proved local solvability for a class of pseudo-differential operators of principal type which need not satisfy condition $(\mathscr{P})$. Egorov's result is the following. Let U be an open set of X and suppose that each point of $p^{-1}(0)$ above U has an open conic neighbourhood Γ in $T^*X \setminus 0$ in which one of the following conditions is fulfilled:

(a) There exists in Γ a C^∞ function μ, homogeneous of degree $(m-1)$, such that $i^{-1}\{\bar{p}, p\} \leqq 2\,\mathrm{Re}\,(\mu p)$ in Γ (here $\{\bar{p}, p\}$ is the Poisson bracket of $\bar{p}$ and p and its expression in local coordinates is

$$\sum_1^n \left(\frac{\partial \bar{p}}{\partial \xi_j} \frac{\partial p}{\partial x_j} - \frac{\partial \bar{p}}{\partial x_j} \frac{\partial p}{\partial \xi_j} \right).$$

(b) For some $z \in \mathbf{C} \setminus \{0\}$, $(\partial/\partial\xi)\,\mathrm{Re}\,(zp) \neq 0$ in Γ and $\mathrm{Im}\,(zp) \geqq 0$ in Γ.

(c) For some $z \in \mathbf{C} \setminus \{0\}$, $(\partial/\partial\xi)\,\mathrm{Re}\,(zp) \neq 0$ in Γ and there exists a conic submanifold Σ of Γ, of codimension 1, such that each null bicharacteristic strip of $\mathrm{Re}\,(zp)$, emanating from a point of Γ, intersects Σ exactly at one point and transversally. Furthermore one assumes that for each null bicharacteristic strip γ of $\mathrm{Re}\,(zp)$ in Γ, one has $\mathrm{Im}\,(zp) \geqq 0$ on γ^- and $\mathrm{Im}\,(zp) \leqq 0$ on γ^+. Here γ^- and γ^+ are defined as follows: Let ϱ_γ be the intersection point of $\gamma \cap \Gamma$ and Σ; if s is the parameter of $\gamma \cap \Gamma$ occuring in the Hamilton—Jacobi equations $dx/ds=$

* Supported partly by a grant from the F. N. R. S. (Belgium) and partly by a grant from the Swedish Institute.