Random linear functionals and subspaces of probability one

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1. Introduction

We start with several definitions.

Let E be a *real, locally convex Hausdorff vector space* (l.c.s.) and $(\Omega, \mathscr{F}, R)$ a probability space. Denote by $\mathscr{B}'$ the least σ-algebra of subsets of the topological dual E' of E, which makes every weakly continuous linear functional on E' measurable. A measurable mapping X of $(\Omega, \mathscr{F})$ into $(E', \mathscr{B}')$ will be called a *random continuous linear functional* (r.c.l.f.) over E. The distribution law of X is written P_X or PX^{-1}. It seems convenient to write

$$\langle X(\omega), \varphi\rangle = X_\varphi(\omega), \quad \omega\in\Omega, \quad \varphi\in E.$$

The characteristic function $\mathscr{L}_X$ of X is defined by

$$\mathscr{L}_X(\varphi) = \mathscr{E}(e^{iX_\varphi}), \quad \varphi\in E.$$

Here $\mathscr{E}$ denotes expectation, that is integration with respect to P.

Two r.c.l.f.'s over E are said to be equivalent (abbr. $\equiv$) if they have the same characteristic function (or distribution law).

Suppose E and F are l.c.s.'s and $\Lambda: E\to F$ a linear continuous mapping. Then for every r.c.l.f. Y over F we get an r.c.l.f. X over E by setting $X={}^t\Lambda\circ Y$. For short, we shall write $X={}^t\Lambda Y$.

The class of all (centred) Gaussian r.c.l.f.'s over E is denoted by $\mathscr{G}(E)(\mathscr{G}_0(E))$.

An r.c.l.f. X over E is said to belong to the class $\mathscr{M}_s(E)$, if for every $\varphi_1, \dots, \varphi_n\in E$, and every $n\in\mathbf{Z}_+$, the distribution law P_Y, $Y=(X_{\varphi_1}, \dots, X_{\varphi_n})$, fulfils the inequality

$$P_Y(\lambda A+(1-\lambda)B) \geqq \big(\lambda P_Y^s(A)+(1-\lambda)P_Y^s(B)\big)^{1/s} \tag{1.1}$$

for every $0<\lambda<1$, and all Borel sets A and B in $\mathbf{R}^n$. Here $s\in[-\infty, 0]$. An r.c.l.f. belonging to the class $\mathscr{M}_{-\infty}(E)$ is called a convex r.c.l.f. over E. Note that $\mathscr{G}(E)\subseteqq \subseteqq\mathscr{M}_0(E)$ [4, Th. 1.1].