On the convergence almost everywhere of double Fourier series

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Let

$$S_{mn}f(x) = \sum_{k=-m}^{m} \sum_{l=-n}^{n} c_{kl} e^{i(kx_1+lx_2)}$$

denote the partial sums of the Fourier series of a function $f \in L^1(\mathbf{T}^2)$ where $\mathbf{T}=[0, 2\pi]$. It was proved by C. Fefferman [4], P. Sjölin [7] and N. R. Tevzadze [8] that if $p>1$ and $f \in L^p(\mathbf{T}^2)$, then $\lim_{n\to\infty} S_{nn}f(x)$ exists almost everywhere. The method of Fefferman and Tevzadze also shows that if $(m_k)_{k=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ are non-decreasing sequences of integers which tend to infinity and $f \in L^2(\mathbf{T}^2)$, then $\lim_{k\to\infty} S_{m_k n_k} f(x)$ exists almost everywhere.

Fefferman [5] also constructed a counterexample which shows that there exists a continuous function f with period 2π in each variable such that $\lim_{m,n\to\infty} S_{mn}f(x)$ exists nowhere. In [7] Sjölin proved that if

$$\sum_{m,n} |c_{mn}|^2 (\log(\min(|m|, |n|)+2))^2 < \infty, \tag{1}$$

then $\lim_{m,n\to\infty} S_{mn}f(x, y)$ exists almost everywhere. From (1) convergence conditions involving the modulus of continuity of f can be obtained. For continuous functions f with period 2π in each variable we set

$$\omega(f; \delta) = \sup_{|x-y|\leqq\delta} |f(x)-f(y)|.$$

It is then known that if

$$\omega(f; \delta) = O\big((\log \delta^{-1})^{-1-\varepsilon}\big), \quad \delta \to 0, \tag{2}$$

for some $\varepsilon>0$, then (1) holds (see Bahbuh [1]). On the other hand it can be proved by use of Fefferman's counterexample that there exists an f with $\omega(f; \delta)=$