Some estimates for spectral functions connected with formally hypoelliptic differential operators

NILS NILSSON

University of Lund, Lund, Sweden

1. Introduction

We are going to consider a differential operator $a(x, D) = \sum a_\alpha(x)D^\alpha$ in and open connected subset S of R^n, where D is the differentiation symbol $(2\pi i)^{-1}(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and the summation is made over a finite number of multi-orders $\alpha = (\alpha_1, \ldots, \alpha_n)$. We assume that the operator $a(x, D)$ is formally hypoelliptic (FHE) of type P in S, i.e. that the (complex-valued) coefficients a_α are in $C^\infty(S)$ and that for every $x \in S$ the polynomial (in $\xi \in R^n$) $|a(x, \xi) = \sum a_\alpha(x)\xi^\alpha$ is equally strong as the hypoelliptic polynomial P in the sense of Hörmander [4]. We also require that the type polynomial P is not a constant. Moreover, we suppose that $a(x, D)$ is formally self-adjoint in S, i.e. that we have $\sum a_\alpha(x)D^\alpha = \sum D^\alpha \overline{a_\alpha(x)}$ there. Then with no loss of generality we may assume that $\operatorname{Re} a(x, \xi) \to +\infty$ as $|\xi| \to \infty$, $\xi \in R^n$, for every $x \in S$ (Lemma 3).

Suppose now that A is a self-adjoint realization of $a(x, D)$ in the Hilbert space $L^2(S)$ (note that A need not be bounded from below), and let $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$ be its spectral resolution, the E_λ being orthogonal projections in $L^2(S)$, increasing with λ. We shall then prove (Theorem 1) that for every real number λ the projection E_λ is given by a kernel e_λ in $C^\infty(S \times S)$ (e_λ is called the spectral function of A) and that, when $\lambda \to -\infty$, e_λ tends exponentially to zero together with its derivatives (with respect to the variables in $S \times S$), uniformly on compact subsets of $S \times S$.

Further, for an arbitrary n-order α we shall investigate the behaviour as $\lambda \to +\infty$ of the derivative $e_\lambda^{(\alpha, \alpha)}(x, y) = D_x^\alpha(-D_y)^\alpha e_\lambda(x, y)$ when $x = y \in S$. We shall then compare $e_\lambda^{(\alpha, \alpha)}(x, x)$ to the function

$$e_{x, \lambda}^{(\alpha, \alpha)}(x, y) = \int_{\operatorname{Re} a(x, \xi) \leq \lambda} \xi^{2\alpha} \exp(2\pi i \langle x - y, \xi \rangle) d\xi .$$