Best uniform approximation by analytic functions

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Introduction

Let $F \in L^p(-\pi, \pi)$, $1 \leq p \leq \infty$, and consider the extremal problem $\inf \|\bar{F} + f\|_p$, f in H^p. For notations and basic results on H^p-spaces, see [6]. This problem was extensively treated by Rogosinski and H. S. Shapiro in [10] and also by Havinson in a series of papers. Havinson studied the corresponding problem in general domains. We refer to the survey [13] by Toumarkine and Havinson, which also contains a quite complete bibliography.

The case $p = \infty$ is of a special interest. It can also be formulated as a problem on so called Hankel matrices and is in this way of importance in probability. We wish to mention in particular the papers by Nehari [9], Hartman [4], Helson and Szegö [5] and Adamyan, Arov and Krein [1]. We shall here consider *continuous* F and look for results on the best approximation f, whereas in the above mentioned papers (except [5]) the results are expressed in terms of the matrix. It is easy to see that in this case a unique best approximation f exists in H^∞. One might ask to what extent do F and f have the same regularity. The investigation of these problems is the main object of the present paper. We shall see that the answer is about the same as for conjugate functions. We shall also restrict ourselves to the case of the unit disc U. However, the function-theoretic proofs in sections 2, 3 and 4 are all of a local character, and so all the results can easily be carried over to any region which has in each case a sufficiently regular boundary.

In section 1 we have stated the dual problem and collected some well-known material. Theorem 1 is originally due to Bonsall [3] and Shapiro [11]. Section 2 is devoted to the study of the extremal functions of the dual problem in H^1. In the case $F \in L^\infty$, de Leeuw and Rudin [8] have examined the question of uniqueness for the corresponding extremal function in H^1. We give a complete solution of this problem, provided F is in C. In section 3 we treat our main problem and in section 4 we give an example which shows that the conditions in Theorem 3 a can not be weakened as long as the regularity of F is expressed only in terms of