Extremal Analytic Functions in the Unit Circle

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1. Introduction

Let u be subharmonic in $\{|z| < 1\}$ and put $A(r) = \inf_{|z|=r} u(z)$, $B(r) = \max_{|z|=r} u(z)$ when $0 < r < 1$. Let λ be a fixed number in (0, 1) and suppose that

$$A(r) \leq \cos \pi\lambda B(r), \quad 0 < r < 1 \tag{1.1}$$

$$u \leq c < +\infty . \tag{1.2}$$

Here c is a positive constant. Under these assumptions Hellsten, Kjellberg and Norstad [4] proved

THEOREM A. *There exists a subharmonic function*

$$U(z) = \frac{2c}{\pi} \tan\left(\frac{\pi\lambda}{2}\right) \operatorname{Re} \int_0^z \frac{t^{\lambda-1} - t^{1-\lambda}}{1 - t^2}\, dt, \quad |\arg z| \leq \pi ,$$

in $\{|z| < 1\}$ *for which* (1.1) *holds with equality and such that for* $0 < r < 1$

$$B(r) \leq U(r) \leq \frac{2c}{\pi\lambda} \tan\left(\frac{\pi\lambda}{2}\right) r^{\lambda} .$$

We note that the author [5] and Essén [1] have proved related theorems. Here we consider a similar problem for analytic functions. More specifically, let σ be a step function on [0, 1], i.e.; a piecewise constant function with a finite number of jumps, which satisfies

(*) σ is upper semicontinuous,

(**) $-1 < \sigma(r) < 1$ when $0 \leq r \leq 1$.