Paley-Wiener type theorems for a differential operator connected with symmetric spaces

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1. Introduction

This paper deals with spectral decomposition of the following differential operators

$$\omega_{p,q}: d^2/dt^2 + g \cdot d/dt$$

where $g(t) = p \cdot \coth t + 2q \cdot \coth 2t$, with p and q positive real numbers and t contained in the open interval $]0, \infty[$.

The radial part of the Laplace-Beltrami operator on a symmetric space of rank one is of the form $\omega_{p,q}$. Here (p, q) are certain pairs of non-negative integers. See Harish-Chandra [11], p. 302, Araki [1].

The main result in this paper is Theorem 4, which generalizes the classical Paley-Wiener theorem, characterizing the Fourier transform of C^∞-functions with compact support, and the theorem that the set of Schwartz-functions is mapped onto itself by the Fourier transform. The results in Theorem 4 (i) and (ii) are well known for symmetric space, see Gangolli [8], Harish-Chandra [11], [12], Helgason [14], [15], Trombi and Varadarajan [23]. However our proof does not use the theory of Lie groups and symmetric spaces at all. Theorem 4 (iii) is probably only known in a special case, see Ehrenpreis and Mautner [6].

The main difficulty in the proof of Theorem 4 is getting the best possible estimates for the eigenfunctions of $\omega_{p,q}$. This is done in Theorem 2. In our proof we use heavily the fact that the eigenfunctions of $\omega_{p,q}$ are essentially hypergeometric functions. It seems, however, that similar results should be obtainable for other differential operators having the same type of singularities as $\omega_{p,q}$. In fact if $p + q < 1$ some results of Dym [5] can be applied to $\omega_{p,q}$ and give a Paley-Wiener theorem which however is weaker than ours. Note also that the classical Hankel-transform is a spectral decomposition of the differential operator $\omega^k = d^2/dt^2 + k \cdot t^{-1} d/dt$,