On the average order of the function $E(x) = \sum_{n \le x} \phi(n) - \frac{3x^2}{\pi^2}$

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1. Introduction

Let n denote a positive integral variable and x denote a real variable, usually ≥ 3. Let $[x]$ denote the greatest integer $\leq x$ and $\{x\} = x - [x]$, called the fractional part of x. Let $\phi(n)$ denote the Euler totient function, which is defined to be the number of positive integers $\leq n$ and relatively prime to n. In 1874, F. Mertens [3] proved that

$$\sum_{n \leq x} \phi(n) = \frac{3x^2}{\pi^2} + O(x \log x) , \tag{1.1}$$

a proof of which may be found in many books on number theory (cf. [2], Theorem 330). It can be easily shown that

$$\sum_{n \leq x} \phi(n)n = \frac{2x^3}{\pi^2} + O(x^2 \log x) . \tag{1.2}$$

Let $\Phi(x)$, $\Phi'(x)$, $E(x)$ and $E'(x)$ be defined by

$$\Phi(x) = \sum_{n \leq x} \phi(n), \quad \Phi'(x) = \sum_{n \leq x} \phi(n)n , \tag{1.3}$$

$$E(x) = \sum_{n \leq x} \phi(n) - \frac{3x^2}{\pi^2} = \Phi(x) - \frac{3x^2}{\pi^2} , \tag{1.4}$$

and

$$E'(x) = \sum_{n \leq x} \phi(n)n - \frac{2x^3}{\pi^2} = \Phi'(x) - \frac{2x^3}{\pi^2} . \tag{1.5}$$