The regularity of growth of entire functions whose zeros are hyperplanes

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Let $f(z)$ be an entire function (of n variables) of finite order ϱ and normal type σ. We then define $h_r(z) = \overline{\lim}_{r\to\infty} r^{-\varrho} \ln|f(rz)|$, $r > 0$ (resp. $h_c(z) = \overline{\lim}_{|u|\to\infty} |u|^{-\varrho} \ln|f(uz)|$, $u \in \mathbf{C}$) and the smallest upper-semicontinuous majorant $h_r^*(z) = \overline{\lim}_{z'\to z} h_r(z')$ (resp. $h_c^*(z) = \overline{\lim}_{z'\to z} h_c(z)$). This is plurisubharmonic and satisfies the condition $h_r^*(tz) = t^\varrho h_r^*(z)$, $t > 0$, (resp. $h_c^*(uz) = |u|^\varrho h_c^*(z)$, $u \in \mathbf{C}$); it is called the radial (resp. circular) indicator function of f.

For $n = 1$, the function $h_r(z)$ is continuous, and so $h_r^*(z) = h_r(z)$ (see [4] or Lemma 1 below), but this is no longer necessarily the case for either $h_r^*(z)$ or $h_c^*(z)$ for $n \geq 2$, [3]. In [1], we undertook a study of the relationship between the distribution of the zeros of $f(z)$ and the local continuity of the function $h_r^*(z)$. We investigate here a condition on the zeros which implies the global continuity of $h_r^*(z)$.

If the function $f(z)$ as a function of several variables depends only upon a single variable, say z_1, and $f(0) \neq 0$, then $h_r^*(z) = h_r(z)$ and the two are continuous. The zeros are then presented by hyperplanes parallel to the hyperplane $z_1 = 0$. We generalize this result in the following way:

THEOREM. *Let $f(z)$ be an entire function of order ϱ and normal type σ such that $f(0) \neq 0$ and the zeros of $f(z)$ are hyperplanes. Then $h_r^*(z) = h_r(z)$ and there are constants T (depending only on σ and ϱ) and α (depending only on ϱ) such that $|h_r(w) - h_r(w')| \leq T\|w - w'\|^\alpha$ for $\|w\| = \|w'\| = 1$. In particular, $h_r^*(z)$ is continuous.*

Remark 1. We will assume, without loss of generality, that we use the Euclidean norm. The value of T depends upon the choice of the norm, but α is independent of the norm chosen.