The asymptotic distribution of the eigenvalues of a degenerate elliptic operator

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1. Introduction

Let R be a Riemannian manifold of dimension $n > 1$ and class C^2, let $\varphi \in C^2(R)$ be real and such that $\varphi = 0 \Rightarrow \operatorname{grad} \varphi \neq 0$ and such that $\varphi \geq 0$ defines a compact part R_φ of R. Let $\Sigma g_{jk} dx^j dx^k$ be the metric of R and $dV = g^{\frac{1}{2}} dx$ $(g = \det (g_{jk}))$ its volume element. Let $L^2(R_\varphi)$ be the real Hilbert space on R_φ with norm square $\int_{R_\varphi} u^2 dV$. Let us interpret the degenerate differential operator

$$\Delta_\varphi = - \sum g^{-\frac{1}{2}} \partial_j \varphi g^{\frac{1}{2}} g^{jk} \partial_k, \quad \partial_j = \partial / \partial x^j \quad (g^{jk}) = (g_{jk})^{-1}$$

as the Friedrichs extension associated with the two quadratic forms

$$a(u) = \int\limits_{R_\varphi} \varphi \sum g^{jk} \partial_j u \partial_k u dV, \quad b(u) = \int\limits_{R_\varphi} u^2 dV$$

and the real space $C^1(R_\varphi)$. According to Baouendi and Goulaouic [1], $A = \Delta_\varphi$ is a non-negative selfadjoint operator on $L^2(R_\varphi)$ and $(I + A)^{-1}$ is compact. Let $\{\lambda_j\}_0^\infty$ be the eigenvalues of A associated with a complete set of eigenfunctions and let $N(\lambda)$ be the number of those eigenvalues which are $\leq \lambda$. We are going to give an asymptotic formula for $N(\lambda)$ as $\lambda \to \infty$. Let dv be the volume element on $S = \partial R_\varphi$ with respect to the induced metric and let $\partial / \partial \nu$ be the unit interior derivative on S. Let ω_n be the volume of the unit ball in R^n and put

$$c_{n-1} = (2\pi)^{1-n} \omega_{n-1} \int\limits_S (\partial \varphi / \partial \nu)^{(1-n)/2} dv \,. \tag{1}$$

Finally, let