Positive harmonic functions vanishing on the boundary of certain domains in $\mathbf{R}^n$

Michael Benedicks

1. Introduction

Let E be a closed, proper subset of the hyperplane $y=0$ in $\mathbf{R}^{n+1}$. A point in $\mathbf{R}^{n+1}$ is, as is customary, denoted by (x, y), where $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$. We assume that each point of E is regular for Dirichlet's problem in $\Omega = \mathbf{R}^{n+1} \setminus E$. C will in the following be a constant, the value of which may vary from line to line.

Consider the cone $\mathcal{P}_E$ of positive harmonic functions in Ω with vanishing boundary values at each point of E. It is easily seen that $\mathcal{P}_E$ contains a non-zero element (Theorem 1).

According to general Martin theory (see e.g. Helms [8]) each positive harmonic function u in an open set Ω may be represented as an integral

$$u(x) = \int_{\Delta_1} K(x, \xi)\, d\mu(\xi),$$

where Δ_1 denotes the set of minimal points in the Martin boundary of Ω. For each $\xi \in \Delta_1$, the function $x \to K(x, \xi)$ is harmonic and minimal positive in the sense of Martin. We recall that a positive harmonic function $u\colon \Omega \to \mathbf{R}$ is minimal positive, if for each positive harmonic function $v\colon \Omega \to \mathbf{R}$

$$v < u \Rightarrow v = \lambda u \quad \text{for some } \lambda, \quad 0 \leqq \lambda < 1.$$

Now we return to the special setting of this paper, i.e. $\Omega = \mathbf{R}^{n+1} \setminus E$, $E \subset \{y=0\}$. In this situation two cases may occur (Theorem 2):

Case 1. All functions in $\mathcal{P}_E$ are proportional.

Case 2. $\mathcal{P}_E$ is generated by two linearly independent, minimal positive harmonic functions.

Stated in terms of Martin theory: the Martin boundary of Ω has either one or two "infinite" points.