On approximation by translates and related problems in function theory

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1. Introduction

From Wiener's approximation theorem we know that the set of finite linear combinations of translates of a function $f \in L(R)$ is dense in $L(R)$ if and only if its Fourier transform is never zero. What can be said if we only allow translates $f(\cdot - \lambda)$ with λ belonging to some fixed set Λ? Problems of this type have been studied by Edwards [3], [4], Ganelius [6], Landau [8], Lönnroth [9], and Zalik [10], [11] among others.

Several approximation problems can be transformed to problems about approximation by translates. We take the Müntz—Szász theorem as an example. Consider approximation in $L(0, 1)$ by linear combinations of monomials x^{μ_k}, where μ_k are distinct numbers greater than -1. Take $g \in L(0, 1)$. Under the transformation $x = \exp(-\exp(-t))$ the expression

$$\int_0^1 \left|\sum a_k x^{\mu_k} - g(x)\right| dx$$

converts to

$$\int_{-\infty}^{\infty} \left|\sum \frac{a_k}{1+\mu_k} f(t - \log(1+\mu_k)) - g(e^{-e^{-t}}) e^{-e^{-t}-t}\right| dt$$

where $f(t) = \exp(-\exp(-t) - t)$. Putting $\lambda_k = \log(1+\mu_k)$ this can be written

$$\int_{-\infty}^{\infty} \left|\sum b_k f(t - \lambda_k) - h(t)\right| dt$$

where $h \in L(R)$.

We will relate the approximation properties of the translates of f to its Fourier transform. In the example above the transform is $\Gamma(1+it) \sim t^{1/2} \exp\left(-\frac{\pi}{2}|t|\right)$, and the corollary to theorem 5 gives the precise answer that approximation is possible