On the regularity of difference schemes

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1. Introduction

In this paper general elliptic difference schemes in Lipschitz regions with Dirichlet boundary conditions are studied. It is shown that the inverse of the difference operator is a uniformly bounded mapping from the analogue of the Sobolev space $H^{\Theta-m}(\Omega)$ onto the analogue of $H_0^{\Theta+m}(\Omega)$ for $|\Theta|<1/2$ ($2m$: order of the differential operator). This property is important for the convergence proof of multi-grid iterations applied to difference schemes, since it is possible to obtain optimal error estimates that are similar to the estimates known from Galerkin approximations. The result is also useful for proving ℓ_∞ stability of difference operators.

Let $\mathbf{Z}$ be the set of all integers, while $\mathbf{Z}_+$ contains all non-negative integers. Following norms will be used for multi-indices $\alpha=(\alpha_1, \ldots, \alpha_d)\in\mathbf{Z}_+^d$ and $\nu=(\nu_1, \ldots, \nu_d)\in\mathbf{Z}^d$:

$$|\alpha| = \alpha_1+\ldots+\alpha_d, \quad \|\nu\| = (\nu_1^2+\ldots+\nu_d^2)^{1/2} \quad (\alpha\in\mathbf{Z}_+^d,\ \nu\in\mathbf{Z}^d).$$

We define the differential operator

$$D^\alpha = i^{-|\alpha|}(\partial/\partial x_1)^{\alpha_1}\ldots(\partial/\partial x_d)^{\alpha_d} \quad (\alpha\in\mathbf{Z}_+^d).$$

Let Ω be a domain in $\mathbf{R}^d$ and consider the boundary value problem

$$Lu = f, \quad u\in H_0^m(\Omega), \tag{1.1}$$

where L is the differential operator

$$(Lu)(x) = \sum\nolimits_{|\alpha|,|\beta|\leqq m} D^\alpha a_{\alpha\beta}(x)D^\beta u(x) \quad (x\in\Omega)$$

of order $2m$. For the notation of the Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$ compare, e.g., [11]. The boundary values are given by $u\in H_0^m(\Omega)$: $(\partial/\partial n)^\nu u(x)=0$ $(0\leqq\nu<m$, $x\in\partial\Omega:=$boundary of Ω, $\partial/\partial n$: normal derivative).

Introduce the regular grid $G_h\subset R^d$ with size h and the grid $\Omega_h\subset G_h$ of Ω by

$$G_h = \{x=\nu h:\ \nu\in\mathbf{Z}^d\}, \quad \Omega_h = G_h\cap\Omega \quad (h\in(0, h_0]).$$