Estimates for maximal functions along hypersurfaces

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1. Introduction

Let $x_{n+1}=F(x_1, \dots, x_n)$ be the equation of a surface in $\mathbf{R}^{n+1}$. We shall study the mean values

$$m_h f(x) = \frac{1}{\prod_1^n h_i} \int_{0<y_i<h_i} f(x'-y, x_{n+1}-F(y))\, dy.$$

Here $h_i>0$, $i=1, \dots, n$, $x=(x', x_{n+1})\in\mathbf{R}^{n+1}$ and $y\in\mathbf{R}^n$. Assuming $F(0)=0$, we ask whether $m_h f\to f$ a.e. as $h_i\to 0$ when $f\in L^p$, $p>1$. This was proved for $F(x')=\prod_1^n x_i^{\alpha_i}$, $\alpha_i>0$, in Carlsson, Sjögren, and Strömberg [1]. Convergence of this type follows from the L^p boundedness of the corresponding maximal function operator

$$M_F f = \sup_{0<h_i<\delta} m_h|f|,$$

where $\delta>0$. Stein and Wainger asked in [2, Problem 8, p. 1289] for which F the operator M_F is bounded on L^p, as a natural extension of the known results for curves. We shall give some answers to this question.

Theorem 1. *Let $F\in C^{2+\varepsilon}$ in a neighborhood of $0\in\mathbf{R}^n$, for some $\varepsilon>0$. If $\partial^2 F(0)/\partial x_i^2\neq 0$, $i=1, \dots, n$, then there exists a δ making M_F bounded on $L^p(\mathbf{R}^{n+1})$, $p>1$.*

Under stronger assumptions on the Hessian of F at 0, the regularity hypothesis can be weakened.

Theorem 2. *Let $F\in C^2$ in a neighborhood of $0\in\mathbf{R}^n$, $n\geqq 2$. Assume that the matrix $(\partial^2 F(0)/\partial x_i\,\partial x_j)_{i,j\in A}$ is nonsingular for any nonempty proper subset A of $\{1, \dots, n\}$. Then M_F is bounded on $L^p(\mathbf{R}^{n+1})$, $p>1$, for some $\delta>0$.*

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