Interpolation by Lipschitz holomorphic functions*

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Introduction

Let $\mathbf{C}^d$ be d-dimensional complex space ($d>1$) with norm $|z|=(|z_1|^2+\ldots+|z_d|^2)^{1/2}$ and unit ball $B=\{z\in\mathbf{C}^d\colon |z|<1\}$. By μ we shall denote the rotation-invariant, normalized Borel measure on $S=\partial B$ and by $C(S)$ — the space of continuous functions on S. If $f\in C(S)$ has a continuous extension $\tilde{f}\colon \bar{B}\to\mathbf{C}$, holomorphic on B, then we shall write $f\in A(B)$. We shall denote $CA=S-A$ for $A\subset S$ and by $[z_1, z_2]$ — any shortest path on S joining z_1 with z_2 ($z_1, z_2\in S$). Let $\varrho(z_1, z_2)$ be the length of a path $[z_1, z_2]$, let $q(z_1, z_2)=|1-\langle z_1, z_2\rangle|$ and let $K(z, r)=\{\xi\in S\colon q(z, \xi)<r\}$ ($\langle z_1, z_2\rangle$ be the scalar product of the vectors z_1 and z_2). We say that $f\in\operatorname{Lip}\alpha$, where $0<\alpha\leqq 1$, if $f\in C(S)$ and there exists a constant C such that

$$|f(z)-f(\xi)| \leqq C\varrho(z, \xi)$$

for $z, \xi\in S$.

Aleksandrov proved [2] that for every real function $g\in C(S)$ and for every $\varepsilon>0$ there exist functions $f\in A(B)$ such that $\operatorname{Re} f\leqq g$ and $\mu(\{z\in S\colon \operatorname{Re} f(z)=g(z)\})\geqq 1-\varepsilon$. Sibony proved [4] that if $f\in A(B)\cap\operatorname{Lip}\alpha$ is a nonconstant function with norm $\|f\|_\infty\leqq 1$, then $\mu(\{z\in S\colon |f(z)|=1\})=0$. This theorem was strengthened by Henkin (see [3] sect. 11.4), who obtained the following result: If $f\in A(B)\cap\operatorname{Lip}\alpha$ is a nonconstant function such that $\operatorname{Re} f\leqq 0$ and $1\geqq\alpha>1/2$, then $\mu(\{z\in S\colon \operatorname{Re} f(z)=0\})=0$. It is still an open problem, if the assumption $1\geqq\alpha>1/2$ can be replaced by a weaker condition $1\geqq\alpha>b$, where $b<1/2$. We shall show that b has to be positive:

Theorem. *For every* $\varepsilon>0$ *there exists* $\alpha>0$ *such that for every real function* $g\in\operatorname{Lip} 1$ *it is possible to find nonconstant functions* $f\in A(B)\cap\operatorname{Lip}\alpha$ *such that* $\operatorname{Re} f\leqq g$ *on* S, *and*

$$\mu(\{z\in S\colon \operatorname{Re} f(z) = g(z)\}) \geqq 1-\varepsilon.$$

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