Wiener's criterion and obstacle problems for vector valued functions

Thomas Karlsson

1. Introduction

The behaviour at the boundary of solutions of the Dirichlet problem in a set $\Omega \subset \mathbf{R}^n$ is a classical problem in the theory for elliptic boundary value problems. In [13] and [14] Wiener considered the case of Laplace's equation. There he gave a geometrical condition, known as *Wiener's criterion for regular boundary points*, which guarantees that solutions attain the boundary values continuously. The condition was given in terms of a series of capacities, measuring the thickness of the complement of Ω, at the point considered. This was generalized to operators with discontinuous coefficients by Littman, Stampacchia, Weinberger [7], and to quasilinear operators by Maz'ja [9] and Gariepy, Ziemer [3]. See also Hildebrandt, Widman [4].

The pointwise continuity is also of interest in the *regularity theory for solutions of obstacle problems*, that is solutions of variational inequalities where the set of admissible variations is given by an obstacle function ψ. In [1] and [2] Frehse and Mosco studied solutions u in a suitable Sobolev space of the variational inequality: $u(x) \geqq \psi(x)$ for $x \in \Omega$ and $\int_\Omega \nabla u \nabla (v-u)\, dx \geqq 0$ for all v in the same Sobolev space with $v(x) \geqq \psi(x)$ for $x \in \Omega$. With an irregular obstacle function ψ they looked at regularity properties at interior points $x_0 \in \Omega$, and one of their results is that solutions are continuous at such points provided a condition of Wiener type is true. Here the condition measures the thickness of certain level sets of ψ at x_0, the meaning of which is precisely described in [1].

The object of this paper is to study *regularity properties of solutions of a class of obstacle problems for vector valued* ($\mathbf{R}^N$-valued, $N \geqq 1$) *functions*, that is when we, instead of one inequality, have a system of inequalities. With a closed and convex set F in $\mathbf{R}^N$, and a closed set E, $E \subset \Omega$, our constraint is of the form $(u-\psi)(x) \in F$ for $x \in E$. Note that in the real case $N=1$, we can for instance choose $F=[0, c]$, $c>0$, and this gives the one-dimensional constraint $\psi(x) \leqq u(x) \leqq \psi(x)+c$ for $x \in E$.