Critical points of Green's function, harmonic measure, and the corona problem

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1. Introduction

Let $\mathscr{R}$ denote a Riemann surface and let $H^\infty(\mathscr{R})$ denote the collection of bounded analytic functions on $\mathscr{R}$. We assume that the functions in $H^\infty(\mathscr{R})$ separate the points of $\mathscr{R}$. Let $\mathscr{M}=\mathscr{M}(H^\infty(\mathscr{R}))$ denote the collection of all complex homomorphisms on $H^\infty(\mathscr{R})$, i.e. $\mathscr{M}$ is the maximal ideal space of $H^\infty(\mathscr{R})$. Each point $\zeta\in\mathscr{R}$ corresponds in a natural way (point evaluation) to an element of $\mathscr{M}$. This paper is concerned with the corona problem for $H^\infty(\mathscr{R})$: Is $\mathscr{M}$ the closure (in the Gelfand topology) of $\mathscr{R}$? More concretely, the problem is:

Given $f_1, \ldots, f_n \in H^\infty(\mathscr{R})$ and $\delta>0$ such that $1 \geqq \max_j |f_j(\zeta)| > \delta$ for all $\zeta\in\mathscr{R}$, is it possible to find $g_1, \ldots, g_n \in H^\infty(\mathscr{R})$ with

$$\sum_{j=1}^n f_j g_j = 1?$$

We refer to $f_1, \ldots, f_n$ as "corona data", $g_1, \ldots, g_n$ as "corona solutions", and $\max \|g_j\|_\infty$ as a "bound on the corona solutions". We reserve n and δ exclusively for the above use. This problem for $\mathscr{U}$, the unit disk in **C**, was conjectured by S. Kakutani in 1941, and solved by L. Carleson [12] in 1962. Not only has the theorem itself been of great interest in classical function theory, but also the proof introduced tools which have been of fundamental importance during the last twenty years. See Garnett's book [25] for an excellent discussion.

After the disk, the next most complicated Riemann surface is an annulus. The simplest proof of the corona theorem in this case is due independently to S. Scheinberg [38] and E. L. Stout [41]. We reproduce it here as motivation for our approach. First one pulls back corona data $\{f_j\}$ on the annulus, A, to functions on a vertical strip S by the map e^z. Since a strip is simply-connected, we may solve

$$1 = \sum f_j(e^z) g_j(z) \quad z\in S,\ g_j \in H^\infty(S) \tag{1.1}$$