Function theory and M-ideals

T. W. Gamelin*, D. E. Marshall*, R. Younis and W. R. Zame*

1. Introduction

This paper deals with some questions that lie at the interface between functional analysis and complex function theory. Let H^∞ denote the algebra of bounded holomorphic functions on the open unit disk. We may identify a function with its radial boundary values, and thus view H^∞ as a subalgebra of the algebra L^∞ of bounded measurable functions on the unit circle. Since H^∞ is a weak-star closed subspace of L^∞, it is easy to see that for every function f in L^∞, there is a function g in H^∞ such that $\|f-g\| = \text{distance}(f; H^\infty)$; i.e., g is a *best approximant* to f from H^∞. Sarason [10] asked whether functions in L^∞ always have best approximants from the space $H^\infty+C$ spanned by H^∞ and the algebra C of continuous functions on the circle.

The space $H^\infty+C$ plays a special role in function theory, since Sarason [9] has shown that it is in fact a closed algebra and is contained in every closed subalgebra of L^∞ that properly contains H^∞. The space $H^\infty+C$ also plays a special role in operator theory, in the following way. Let L^2 denote the Hilbert space of square integrable functions on the circle, let H^2 denote the closed subspace spanned by the non-negative powers of the function z, and let $(H^2)^\perp$ be the closed subspace spanned by the negative powers of z (which is just the orthogonal complement of H^2). We write Q for the orthogonal projection of L^2 onto $(H^2)^\perp$. For each $f \in L^\infty$, we define a *Hankel operator* H_f: $H^2 \to (H^2)^\perp$ by $H_f(g) = Q(gf)$. If we use the usual bases for H^2 and $(H^2)^\perp$, we may characterize Hankel operators as those whose matrices have constant cross diagonals. It turns out that the distance (in the operator norm) from the Hankel operator H_f to the space of all compact operators from H^2 to $(H^2)^\perp$ is the same as the distance from H_f to the space of compact Hankel operators, and the latter coincides with the distance (in the L^∞-norm) from the

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