On maximal functions generated by Fourier multipliers

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1. Introduction

Denote by S the space of all infinitely differentiable, rapidly decreasing functions on $\mathbf{R}^n$ and by $L^p=L^p(\mathbf{R}^n)$, $1\leqq p\leqq\infty$, the standard Lebesgue spaces. Then $m\in L^\infty$ is said to be an M_p-multiplier if

$$\|m\|_{M_p} = \inf\{C\colon \|F^{-1}[mf^\wedge]\|_p \leqq C\|f\|_p,\, f\in S\}$$

is finite. Here $^\wedge$ or F denotes the Fourier transformation and F^{-1} its inverse. The purpose of this paper is to examine maximal functions of the type (A_t being a dilation matrix)

$$M_m f(x) = \sup_{t>0} |F^{-1}[m(A_{1/t}\xi)f^\wedge(\xi)](x)|,\; f\in S, \tag{1.1}$$

from the point of view of the Fourier multiplier m.

If $F^{-1}[m]$ is integrable and has an integrable decreasing radial majorant then it is well known (see e.g. Stein [18; p. 62] for isotropic dilations and for anisotropic ones see Madych [14]) that $M_m f$ can be dominated pointwise by the classical Hardy—Littlewood maximal function:

$$M_m f(x) \leqq CMf(x) \text{ a.e.},\; Mf(x) = \sup_{r>0} r^{-n}\int_{|y|\leqq r} |f(x-y)|\,dy. \tag{1.2}$$

(We use C as a constant, independent of f and x, which is not necessarily the same at each occurrence.) Moreover, for radial integrable $F^{-1}[m]$ there holds

$$\|M_m f\|_p \leqq C\|f\|_p \tag{1.3}$$

provided $n\geqq 3$ and $p>n/(n-1)$ which is a consequence of Stein's [20] result on spherical means. For the case $p=n=2$, Aguilera [1] has given sufficient conditions on radial $F^{-1}[m]$ to satisfy (1.3).