On the asymptotics of solutions of Volterra integral equations

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1. Introduction

It is well-known (see e.g. [1]) that a solution of a Volterra integral equation

$$v(t)-\int_{t_0}^{\infty} G(t,\tau)v(\tau)\,d\tau = v_0(t) \tag{1.1}$$

exists, is unique, and may be obtained by a series of iterations

$$v=\sum_{n=0}^{\infty} v_n, \quad v_{n+1}(t)=\int_{t_0}^{t} G(t,\tau)v_n(\tau)\,d\tau. \tag{1.2}$$

Moreover, if the kernel $G(t,\tau)$ vanishes sufficiently quickly (in a power scale) for large values of the variables, then the functions $v_n(t)$ can be bounded by a common power of t (for a polynomially bounded free term v_0). It follows that the solution $v(t)$ of (1.1) is bounded by the same power of t. On the other hand, if $G(t,\tau)$ decays slowly (or grows) as $t,\tau\to\infty$, then generally $v_{n+1}(t)\,v_n(t)^{-1}\to\infty$, $t\to\infty$. In this case the series (1.2) gives only a bound of exponential type for $v(t)$ (even if $v_0(t)$ has compact support).

In the present paper we study the behaviour of the solution of the equation (1.1) with a slowly decreasing (or growing) kernel $G(t,\tau)$. This problem is close in spirit to the investigation of the asymptotics of solutions to differential equations. Consider, for example, the simplest equation

$$-v''(t)+q(t)v(t)=0, \quad t\geqq t_0. \tag{1.3}$$

If the function $q(t)$ has compact support, then the equation (1.3) has solutions that equal 1 or t for large t. Similarly, if $q(t)=0(t^{-2-\varepsilon})$, $\varepsilon>0$, then solutions of (1.3) approach 1 or t asymptotically. The proof of this assertion may be obtained by reduction of (1.3) to a Volterra integral equation. However, if $q(t)$ decays slower than t^{-2}, or grows as $t\to\infty$, then the terms $v_n(t)$ of the corresponding series (1.2) obey the relation $v_{n+1}(t)v_n(t)^{-1}\to\infty$, $t\to\infty$. This changes the asymptotics of $v(t)$. In such a case the asymptotics of $v(t)$ was found by Greene and Liouville (see e.g. [2]) with