On the snow flake domain

Robert Kaufman and Jang-Mei Wu

Introduction

We show that *on the boundary of the snow flake domain, harmonic measure lies completely on a set of Hausdorff dimension less than that of the entire boundary.* It is surprising because the snow flake domain is highly symmetric.

Øksendal has proved that in the plane, harmonic measure is always singular with respect to area measure, and conjectured that harmonic measure is singular with respect to α-dimensional Hausdorff measure for any $\alpha > 1$, see [4]. Our example hints that his conjecture may be true.

According to Lehto and Virtanen [3], the construction of the snow flake domain is due to G. Piranian.

Let T_0 be an equilateral triangle with side length 1 and center P. We subdivide each side of T_0 into three equal subintervals of length 1/3 each; and build an equilateral triangle over each middle subinterval, exterior to T_0 and with one side coinciding with that interval; call these triangles $S_{1,i}$, $i=1, 2, 3$ and let $T_1 = T \cup \bigcup_{i=1}^{3} S_{1,i}$.

Suppose T_j has been constructed, and is a polygon with 3×4^j sides each, of side length 3^{-j}. We subdivide each side of T_j into three equal subintervals of length 3^{-j-1} each; and build an equilateral triangle over each middle subinterval, exterior to T_j and with one side coinciding with that subinterval; call these triangles $S_{j+1,i}$, $1 \leqq i \leqq 3\times 4^{j+1}$ and let $T_{j+1} = T_j \cup \bigcup_{i=1}^{3\times 4^{j+1}} S_{j+1,i}$. Let

$$T = \bigcup_{j=1}^{\infty} T_j, \quad \bar{\Omega} = \bar{T} \quad \text{and} \quad \Omega = \bar{\Omega}^0.$$

Let f be a homeomorphism from $\partial\Omega$ onto [0, 3] (mod 3), such that the three vertices of T_0 are mapped to 1, 2, $3 \equiv 0$ (mod 3), and any two endpoints of a side of T_j are mapped to points on [0, 3] of distance 4^{-j} to each other. Since vertices of T_j are in $\partial\Omega$, we identify vertices of T_j with points in [0, 3] (mod 3) by their quaternary expansion whenever it is convenient to do so. The vertices of T_j are exactly those points in $\partial\Omega$ whose quaternary expansion terminates.