Phragmén—Lindelöf's and Lindelöf's theorems

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1. Introduction

Both Phragmén—Lindelöf's and Lindelöf's theorems consider behavior of a function at a boundary point and, originally, cf. [PL], [L], the proofs for these theorems employ harmonic measure. In local properties the measure theoretic aspect of harmonic measure plays a minor role and in this paper we show that even for non-linear partial differential equations and quasiregular mappings it is possible to prove corresponding results using so called F-harmonic measure, which is intimately connected with the corresponding differential equation or variational integral, cf. [GLM2].

We shall study the conformally invariant case, i.e. we consider extremals of the variational integral

$$\int F(x, \nabla u)\, dm,$$

where $F(x, h) \approx |h|^n$ and n is the dimension of the Euclidean space $\mathbf{R}^n$. Thus the plane harmonic case is included but the classical harmonic case in space $\mathbf{R}^n$, $n \geqq 3$, is not. In general, our methods only work in the "borderline" case $F(x, h) \approx |h|^n$.

The proofs for Phragmén—Lindelöf's theorem in domains more general than sectors usually combine the method invented by T. Carleman [C], cf. also [T, Theorem III. 67], with a principle which we call Phragmén—Lindelöf's principle. This is a slight misuse of the name, cf. e.g. [A, p. 40]. The principle, Theorem 3.5, relates in classical terms the growth of a harmonic function with the density of a harmonic measure at ∞. The density concept extends to the non-linear case and hence the principle holds in the more general situation even in a sharp form. Carleman's method is based on the study of the Carleman mean

$$\int_{G \cap S^1(t)} u^2\, ds$$

of a suitably chosen harmonic measure u. For a good account of the development in the field see [Ha]. Via Wirtinger's inequality [T, p. 112] the Carleman mean can be