On the multiplicative properties of the de Rham-Witt complex. II.

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Introduction

The purpose of this paper is to present a Künneth formula for the cohomology of the de Rham—Witt complex (cf. [15]) for smooth and proper varieties over a perfect field k. To explain the details of this let me recall the form the Künneth formula takes for the crystalline cohomology. In this case the cohomology of a smooth and proper variety X over k is a certain complex of W-modules $R\Gamma(X/W)$ where W is the ring of Witt vectors of k. The Künneth formula then takes the form $R\Gamma(X/W)\otimes_W^L R\Gamma(Y/W)=R\Gamma(Xx_kY/W)$. The reason for the appearance of the tensor product is that it is the product universal for the properties of the cup product in crystalline cohomology namely W-bilinearity. The Künneth formula is proved by first, with the aid of the cup product, defining a morphism $R\Gamma(X/W)\otimes_W^L R\Gamma(Y/W)\to R\Gamma(Xx_kY/W)$ and then noting that as both sides of this morphism are complexes whose cohomology are finitely generated W-modules to prove that it is an isomorphism it suffices to show that $W/p\otimes_W^L(-)$ applied to it is. Finally, $W/p\otimes_W^L((-)\otimes_W^L(-))=(W/p\otimes_W^L(-))\otimes_W^L(W/p\otimes_W^L(-))$ and $W/p\otimes_W^L R\Gamma(Z/W)\doteq H_{DR}(Z/k)$ for any smooth k-variety Z. That the Künneth morphism is an isomorphism is now clear as the Künneth morphism in de Rham-cohomology is. From our point of view this proof has one drawback. It needs the rather precise piece of information that $R\Gamma(Z/W)$ has finitely generated cohomology when Z is a smooth and proper variety over k. This may be rectified as follows. For a general morphism $M\to N$ in the derived category of W-complexes it is true that if $W/p\otimes_W^L(-)$ applied to it is an isomorphism then its completion also is, where the completion functor, $(\hat{-})$, is defined to be the composite $R\varprojlim\{W/p^n\otimes_W^L, \pi\}$, π being the morphisms induced by projections $W/p^{n+1}\to W/p^n$. We hence get a Künneth formula involving the completed tensor product, $((-)\otimes_W^L(-))\hat{}$, this time there being no need to assume that X and Y be proper. This version of the Künneth formula is then completed by a calculation of $(M\otimes_W^L N)\hat{}$