# Spaces of Besov—Hardy—Sobolev type on complete Riemannian manifolds

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## 1. Introduction

Let $\mathbf{R}_n$ be the Euclidean $n$-space. The two scales of spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^{s}_{p,q}(\mathbf{R}_n)$ with $-\infty<s<\infty$, $0<p\leqq\infty$ ($p<\infty$ in the case of the $F$-spaces), $0<q\leqq\infty$, cover many well-known classical spaces of functions and distributions on $\mathbf{R}_n$:

(i) the Besov—Lipschitz spaces $\Lambda^s_{p,q}(\mathbf{R}_n)=B^s_{p,q}(\mathbf{R}_n)$ if $s>0$, $1<p<\infty$, $1\leqq q\leqq\infty$;

(ii) the Bessel-potential spaces $H^s_p(\mathbf{R}_n)=F^s_{p,2}(\mathbf{R}_n)$ if $-\infty<s<\infty$, $1<p<\infty$, with the Sobolev spaces $W^m_p(\mathbf{R}_n)=H^m_p(\mathbf{R}_n)$ if $1<p<\infty$, $m$ non-negative integer, as special cases;

(iii) the Hölder—Zygmund spaces $\mathscr{C}^s(\mathbf{R}_n)=B^s_{\infty,\infty}(\mathbf{R}_n)$ if $s>0$;

(iv) the (non-homogeneous) Hardy spaces $H_p(\mathbf{R}_n)=F^0_{p,2}(\mathbf{R}_n)$ if $0<p<\infty$.

After a modified extension of the definition of $F^0_{p,2}(\mathbf{R}_n)$ to $p=\infty$ one can even include the non-homogeneous version of the fashionable space BMO of functions of bounded mean oscillation.

The definition of the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ is based on decompositions of the Fourier image of tempered distributions on $\mathbf{R}_n$. It is due to J. Peetre, P. I Lizorkin and the author, cf. [14—17, 25] (a more careful description of these historical aspects may be found in [29, 2.3.5]). Systematic treatments of these spaces have been given in [18] (mostly restricted to $B^s_{p,q}(\mathbf{R}_n)$ with $1\leqq p\leqq\infty$) and [29] (with [26, 27] as forerunners, cf. also [28]). The problem arises to introduce spaces of $B^s_{p,q}-F^s_{p,q}$ type on manifolds, where (complete) Riemannian manifolds and Lie groups seem to be of peculiar interest. As far as special cases of the above spaces are concerned something has been done in this direction. T. Aubin [1, and Chapter 2 in 2] studied Sobolev spaces on (complete) Riemannian manifolds, which are defined via covariant derivatives. Weighted Sobolev spaces on Riemannian manifolds may be found in [4]. Based on a study of the Laplace—Beltrami operator R. S. Strichartz [24] introduced Bessel potential spaces on complete Riemannian manifolds. Lipschitz—Besov—Hardy spaces on Lie groups attracted even more attention.