Wiener's theorem, the Radon—Nikodym theorem, and $m_0(\mathbf{T})$

Russell Lyons

1. Introduction

Let $M(\mathbf{T})$ denote the class of complex Borel measures on the circle $\mathbf{T}=\mathbf{R}/\mathbf{Z}$ and $M_0(\mathbf{T})$ the subclass $\{\mu\colon \lim_{n\to\infty}\hat{\mu}(n)=0\}$. It was recently proved [5, 6] that $M_0(\mathbf{T})$ is characterized by its class of common null sets. To make this more precise, we use the following notation. For any subclass $\mathscr{C}\subset M(\mathbf{T})$, we let

$$\mathscr{C}^{\perp} = \{E\subset T\colon E \text{ is a Borel set and } \forall\mu\in\mathscr{C}\ |\mu|(E) = 0\}$$

be the class of common null sets of $\mathscr{C}$. Likewise, if $\mathscr{E}$ is a class of Borel subsets of $\mathbf{T}$, we write

$$\mathscr{E}^{\perp} = \{\mu\in M(\mathbf{T})\colon \forall E\in\mathscr{E}\,|\mu|(E) = 0\}$$

for the class of measures annihilating $\mathscr{E}$. Then by definition, the class of sets of uniqueness in the wide sense, U_0, is equal to $M_0(\mathbf{T})^{\perp}$ and [6] shows that $U_0^{\perp}=M_0(\mathbf{T})$. That is, $M_0(\mathbf{T})^{\perp\perp}=M_0(\mathbf{T})$.

Now notice that we can write $M_0(\mathbf{T})$ in another way. Let PM be the pseudo-measure topology on $M(\mathbf{T})$: $\|\mu\|_{PM}\equiv\sup_{n\in\mathbf{Z}}|\hat{\mu}(n)|$. If $\mathscr{P}$ denotes the trigonometric polynomials and λ Lebesgue measure on $\mathbf{T}$, then $M_0(\mathbf{T})$ is the PM-closure of $\mathscr{P}.\lambda$.

If M denotes the usual norm topology on $M(\mathbf{T})$, then the M-closure of $\mathscr{P}.\sigma$, for any $\sigma\in M(\mathbf{T})$, is $L^1(\sigma)=\{f.\sigma\colon \int|f|\,d\,|\sigma|<\infty\}$. It is clear that $L^1(\sigma)^{\perp}=\{E\colon |\sigma|(E)=0\}$, whence the Radon—Nikodym theorem is equivalent to the assertion $L^1(\sigma)^{\perp\perp}=L^1(\sigma)$. This leads us to ask if the analogous theorem holds for PM. In other words, if $L^{PM}(\sigma)$ denotes the PM-closure of $\mathscr{P}.\sigma$, is $L^{PM}(\sigma)^{\perp\perp}=L^{PM}(\sigma)$?

Consider now Wiener's theorem [3, p. 42], which says that for all $\mu\in M(\mathbf{T})$,

$$V(\mu) \equiv \lim_{N\to\infty}\left(\frac{1}{2N+1}\sum_{|n|\leq N}|\hat{\mu}(n)|^2\right)^{1/2} \tag{1}$$