On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles

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Introduction

The cohomology groups of the moduli space of complex vector bundles of coprime rank n and degree d over a compact Riemann surface M have been computed by Harder and Narasimhan [H & N] and also by Atiyah and Bott [A & B]. The basic idea of Atiyah and Bott is to apply equivariant Morse theory to the Yang—Mills functional on the infinite-dimensional affine space of connections on a fixed C^∞ bundle. However they avoid the analytic problems involved in infinite-dimensional Morse theory by giving an alternative definition of the Morse stratification in terms of the canonical filtrations introduced by Harder and Narasimhan. This stratification turns out to be equivariantly perfect relative to the gauge group of the bundle, which means that the equivariant Morse inequalities are in fact equalities. These then provide an inductive formula for the equivariant cohomology of the minimal or semistable stratum, from which the cohomology of the moduli space can be calculated.

Atiyah observed that the same idea could be applied to the action of a complex reductive group G on a finite-dimensional nonsingular complex projective variety X. Such a variety has a symplectic structure which is preserved by a maximal compact subgroup K of G. To this symplectic action there is associated a moment map. The Yang—Mills functional can be regarded as an analogue of the norm-square of the moment map. In [K] it is shown that the norm-square of the moment map always induces an equivariantly perfect stratification of X. In good cases when every semistable point of X is stable, this stratification can be used to obtain a formula for the Betti numbers of the geometric invariant theory quotient of X by G.

One method of constructing the moduli spaces of vector bundles over M is to identify them with quotients of certain finite-dimensional quasi-projective varieties,