H^∞+BUC does not have the best approximation property

Carl Sundberg

§ 1. Introduction

Let L^∞ denote the usual Lebesgue space of functions on the unit circle $[|z|=1]$ and let H^∞ denote the bounded analytic functions on the unit disc $[|z|<1]$. By identifying functions in H^∞ with their boundary values we may regard H^∞ as a closed subalgebra of L^∞. The closed algebras between H^∞ and L^∞ are called *Douglas algebras* and have been studied extensively ([3], [4], [5], [9], [11], [14], [15]). For background and general information on Douglas algebras see [6] and [13].

Let C denote the space of continuous functions on the unit circle. It was shown by Sarason [10] that the linear span $H^\infty+C$ is a Douglas algebra. In fact it is the smallest such algebra properly containing H^∞; see [7]. In [12], Sarason asked whether $H^\infty+C$ has the *best approximation property,* i.e. whether given any $f\in L^\infty$ there existed a $g\in H^\infty+C$ such that

$$\|f-g\|_\infty = \mathrm{d}(f, H^\infty+C) \underset{\mathrm{def}}{=} \inf\{\|f-g\|_\infty:\ g\in H^\infty+C\}.$$

This question was answered affirmatively by Axler, Berg, Jewell, and Shields [1], who then raised the question of whether all Douglas algebras possess this property.

A subsequent paper of Luecking [8] provided a simpler proof of the $H^\infty+C$ case using the theory of M-ideals. In an unpublished manuscript, Marshall and Zame give a very simple proof of this case and also give many interesting examples of Douglas algebras possessing the best approximation property. Another such example is given by Younis in [16].

In this paper we answer the question for general Douglas algebras negatively, our counterexample being a certain "natural" Douglas algebra. In order to describe and work with this algebra it is convenient to move over to the real line $\mathbf{R}$ and the upper half plane $\Delta=\{z=x+iy\colon x, y\in\mathbf{R}, y>0\}$. Henceforth in this paper L^∞ and H^∞ will refer to the corresponding function spaces on $\mathbf{R}$ and Δ. Let BUC