On a differential equation arising in a Hele Shaw flow moving boundary problem

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1. Introduction

The present paper is mainly devoted to the following differential equation:

Given $f(\zeta)$, *analytic and univalent in a neighbourhood of* $|\zeta|\leqq 1$, *find* $f(\zeta, t)$, *analytic and univalent as a function of* ζ *in a neighbourhood of* $|\zeta|\leqq 1$, *continuously differentiable with respect to* t *for* $t\in\mathbf{R}$ *in an interval containing* $t=0$, *satisfying*

$$\operatorname{Re}[\dot{f}(\zeta, t)\cdot\overline{\zeta f'(\zeta, t)}] = 1 \quad \text{for} \quad |\zeta| = 1 \tag{1}$$

and $f(\zeta, 0)=f(\zeta)$ (*for* $|\zeta|<1$). *In* (1) $\dot{f}$ *and* f' *denote derivatives with respect to* t *and* ζ *respectively.*

This differential equation arose in the paper [5] by S. Richardson as describing the solution of a two-dimensional moving boundary problem for so called Hele Shaw flows. The moving boundary in question then was the boundary of the domain $\Omega_t=f(\mathbf{D}, t)$, where $\mathbf{D}=\{\zeta\in\mathbf{C}: |\zeta|<1\}$ and t is time. Richardson did not prove existence or unicity for solutions of (1). However, this, essentially, was done in [10]. The existence of solutions was proved by using an iterative process, the proof of convergence of which was fairly complicated. Unicity was proved only with respect to solutions which depended analytically on t.

The aim of the present paper is primarily to give a more elementary proof of existence of solutions of (1) in the case that $f(\zeta)$ is a polynomial or a rational function. In that case (1) can be reduced to a finite system of ordinary differential equations (in t) and this system has a unique solution by standard theory. This solution is a polynomial or a rational function (as a function of ζ) of the same sort as $f(\zeta)$. (Theorem 4.)

We will also consider a generalization of the differential equation (1) in order to prove a result on the "moment map"

$$f \to (c_0, c_1, c_2, \ldots), \tag{2}$$