A superharmonic proof of the M. Riesz conjugate function theorem

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Introduction

Let $f=f+i\tilde{f}$ be analytic in the unit disk U with $\tilde{f}(0)=0$. It is known that

$$\|F\|_p^p \leqq C_p \|f\|_p^p, \quad 1<p<\infty. \tag{0.1}$$

The purpose of this note is to give a simple proof of this theorem of M. Riesz, using superharmonic functions. For the related inequality

$$\|\tilde{f}\|_p^p \leqq C_p' \|f\|_p^p, \quad 1<p<\infty, \tag{0.2}$$

the best constant was determined by S. K. Pichoıides [3] and, independently, by B. Cole (cf. Gamelin [2] p. 144). (The relation between (0.1) and (0.2), with best constants, is discussed in Remark 2 at the end of Section 2.) A related result of Cole is given in Theorem 8.3 in [2]. Our proof will also give the best constant C_p for $1<p<\infty$. We do not use duality to go from the case $1<p<2$ to the case $2<p<\infty$. In Section 3 we discuss similar inequalities for other plane domains.

What the earlier work and our work have in common is the use of sub- or super-harmonic functions. What is new in our approach is how we choose the superharmonic functions.

A similar idea can be found as early as 1935 (cf. Section 4). In Section 5, we use this idea of P. Stein to extend (0.1) to higher dimensions in the case $1<p\leqq 2$.

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1. The case $1<p<2$

Let $w=u+iv$ be a complex variable. If $\alpha=\pi/2p$, we define

$$G(w) = \begin{cases} |w|^p-(\cos\alpha)^{-p}|u|^p, & \alpha < |\arg w| < \pi-\alpha, \\ -\tan\alpha\, |w|^p \cos p\theta, & |\theta| < \alpha, \quad \text{where} \quad \theta = \arg w, \\ -\tan\alpha\, |w|^p \cos p(\pi-|\theta|), & 0 \leqq \pi-|\theta| < \alpha. \end{cases}$$