Somewhat quasireflexive Banach spaces

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The question of "What kind of subspaces must a nonreflexive Banach space X have?" has received a lot of attention. Pelczynski [23] (in 1962) has given the most general answer to date: X contains a basic sequence which is not shrinking (and hence spanning a nonreflexive space). For special cases more is known. Johnson and Rosenthal [8] have shown that X and X^* contain reflexive subspaces if X^{**} is separable. (This was extended to the case when X^{**}/X is separable by Clark [2].) In another direction, Davis and Johnson [5] have shown that if X^{**}/X is infinite dimensional then X contains a basic sequence that spans a nonquasireflexive subspace. Perhaps the main reason for this interest are the following two long open questions:

(1) Does each Banach space contain an unconditional basic sequence?
(2) Does each Banach space contain a subspace isomorphic to c_0, l_1 or to a reflexive space?

Indeed, James [6] has shown that a positive answer to (1) implies a positive answer to (2). And clearly these results are partial answers to (2).

On the other hand, consider the collection of spaces to which the special cases apply. James [7], Lindenstrauss [9], Davis, Figiel, Johnson and Pelczynski [4] and the author [1] show how to construct an X so that X^{**}/X is a pregiven Z (with restrictions on Z). All these constructions depend on reflexivity or quasireflexivity in a strong way and the constructed X has lots of quasireflexive subspaces.

This paper attempts to unite these results. It is shown that if X^{**}/X is separable then each element of X^{**}/X is "reachable" by an order one quasireflexive subspace $Z \subset X$, so that Z has a shrinking basis (Theorem 8). If X^{**} is separable, both X and X^* have subspaces and quotients which are order one quasireflexive with bases (Theorem 9). And if X^* is separable then X has a nonreflexive quotient

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