Area integral estimates for elliptic differential operators with non-smooth coefficients

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In this note we shall prove inequalities comparing the area integral and nontangential maximal functions for solutions to second order elliptic equations in a domain in $\mathbf{R}^n$, in which both the coefficients of the equation and the domain satisfy very weak regularity conditions to be formulated later (cf. [7]). Such inequalities have been proved by many authors in increasing generality. (See [1, 6, 9, 11], where further references can be found.) The most general setting up to now is that of harmonic functions in Lipschitz domains [6]. There the key additional point is the fact that harmonic measure for the standard Laplace operator satisfies A_∞ (a scale-invariant form of mutual absolute continuity) with respect to surface measure on the boundary of the domain. By contrast, surface measure need not exist in our more general context. Moreover, even if the domain is smooth (and hence has a surface measure), L-harmonic measure and surface measure are mutually singular for some choices of the elliptic operator L [2]. This is the main new difficulty.

We shall first state and prove the theorem in a special case that retains the main difficulty. Recall that a bounded domain $D\subset\mathbf{R}^n$ is called a Lipschitz domain if ∂D can be covered by finitely many open right circular cylinders whose bases have positive distance from ∂D and corresponding to each cylinder I there is a coordinate system (x, y) with $x\in\mathbf{R}^{n-1}$, $y\in\mathbf{R}$ with y axis parallel to the axis of I, and a function $\varphi\colon \mathbf{R}^{n-1}\to\mathbf{R}$ satisfying a Lipschitz condition $\left(|\varphi(x)-\varphi(z)|\leqq M|x-z|\right)$ such that $I\cap D=\{(x,y)\colon y>\varphi(x)\}\cap I$ and $I\cap\partial D=\{(x,y)\colon y=\varphi(x)\}\cap I$. Denote by $L=\sum_{i,j=1}^n \frac{\partial}{\partial X_i} a_{ij}(X)\frac{\partial}{\partial X_j}$ a uniformly elliptic operator with bounded, measurable coefficients, that is, $a_{ij}\in L^\infty(\mathbf{R}^n)$ and for some $c>0$, $\sum_{i,j=1}^n a_{ij}(X)\xi_i\xi_j\geqq c|\xi|^2$ for

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