Extension of a result of Benedek, Calderón and Panzone

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1. Introduction

For X a Banach space and $1 \leq p \leq \infty$, L_X^p is the usual Lebesgue space.

The theorem of Benedek, Calderón and Panzone [0] asserts that for $1 < p$, $r < \infty$, any operator T: $L_{l^r}^p(\mathbf{R}^n) \to L_{l^r}^p(\mathbf{R}^n)$ of the form $T(f_j) = P.V.\ (K_j * f_j)$ is bounded, the (K_j) being a sequence of convolution kernels K satisfying the conditions

(a) $\|\hat{K}\|_\infty \leq C$

(b) $|K(x)| \leq C|x|^{-n}$

(c) $|K(x) - K(x-y)| \leq C|y|\,|x|^{-n-1}$ for $|y| < \dfrac{|x|}{2}$

and where C is a fixed constant.

Our purpose is to show that this theorem remains true if one replaces l^r by any lattice X with the so-called UMD-property (cf. [2]). Let us recall that a Banach space X is UMD provided for $1 < p < \infty$ martingale difference sequences $d = (d_1, d_2, \ldots)$ in $L_X^p[0, 1]$ are unconditional, i.e. $\|\varepsilon_1 d_1 + \varepsilon_2 d_2 + \ldots\|_p \leq C_p(X)\|d_1 + d_2 + \ldots\|_p$ whenever $\varepsilon_1, \varepsilon_2, \ldots$ are numbers in $\{-1, 1\}$. This property is also equivalent to the boundedness of the Hilberttransform on $L_X^p(\mathbf{R})$ (see [3], [1]) and can be characterized geometrically by the existence of a symmetric, biconvex function ζ on $X \times X$ satisfying $\zeta(x, y) \leq \|x+y\|$ if $\|x\| \leq 1 \leq \|y\|$ and $\zeta(0, 0) > 0$. Let us point out that also for lattices UMD is more restrictive than a condition of r-convexity, s-concavity for some $1 < r$, $s < \infty$ (see [9]).

Theorem. *Assume X is a* UMD *space with a normalized unconditional basis (e_j). Then, for $1 < p < \infty$, any operator T: $L_X^p(\mathbf{R}^n) \to L_X^p(\mathbf{R}^n)$ defined as*

$$T(\Sigma f_j e_j) = \Sigma T_j(f_j) e_j$$

where the T_j are the singular integral operators considered above, is bounded.