On additive automorphic and rotation automorphic functions

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1. Introduction

Let D denote the unit disc in the complex plane and let Γ be a Fuchsian group. A function $W(z)$ meromorphic in D is said to be *additive automorphic relative to the Fuchsian group* Γ if for each transformation $T \in \Gamma$ there exists a constant A_T such that $W(T(z)) = W(z) + A_T$ for each $z \in D$. The numbers A_T are called *periods* of $W(z)$. A function $W(z)$ is said to be *additive automorphic* if it is additive automorphic relative to some non-trivial Fuchsian group. An analytic function $f(z)$ in D is said to be a *Bloch function* if there exists a constant B_f such that $(1-|z|^2)|f'(z)| \leqq B_f$ for each $z \in D$. A function $f(z)$ meromorphic in D is said to be a *normal function* if there exists a constant N_f such that

$$(1-|z|^2)|f'(z)|/(1+|f(z)|^2) \leqq N_f$$

for each $z \in D$.

Conditions under which an additive automorphic function is a normal function have been studied by Aulaskari [1], [2]. Pommerenke [6] has given an example of an additive automorphic function $W(z)$ such that $W(z)$ is not a Bloch function but $\iint_F |W'(z)|^2 dx\, dy < \infty$, where F denotes the fundamental region for Γ. In this note the main result is the following theorem.

Theorem 1. *There exists an additive automorphic function $W(z)$ relative to a Fuchsian group Γ such that $W(z)$ is not a normal function, $W(z)$ has only imaginary periods, and*

$$\iint_F |W'(z)|^2\, dx\, dy < \infty,$$

where F denotes the fundamental region of Γ.

This theorem will be proved in Section 2, by means of a modification of Pommerenke's method. In Section 3, some important consequences of this theorem will be given.