The extension problem for certain function spaces involving fractional orders of differentiability

M. Christ

1. Introduction

The purpose of this paper is to study the question of extendability to the whole space of functions defined on sub-domains of $\mathbf{R}^n$ and satisfying certain smoothness conditions. The usual Sobolev spaces of integral order are defined by

$$L^p_k(\Omega) = \{f\in L^1_{\mathrm{loc}}(\Omega)\colon D^\beta f\in L^p(\Omega), \quad \text{for all} \quad |\beta| \leqq k\},$$

when Ω is connected, $1\leqq p\leqq\infty$ and $k\in Z^+$; the derivatives are assumed to exist in the sense of distributions on Ω. $\|f\|_{L^p_k(\Omega)}$ is defined to be

$$\sum\nolimits_{0\leqq|\beta|\leqq k}\|D^\beta f\|_{L^p(\Omega)}.$$

By an extension operator for $L^p_k(\Omega)$ we will mean a bounded linear operator $\Lambda\colon L^p_k(\Omega)\to L^p_k(\mathbf{R}^n)$, such that $\Lambda(f)\equiv f$ on Ω. Ω will be called an *extension domain* for L^p_k if such an extension operator exists.

Calderon [4] showed that if $\partial\Omega$ is locally the graph of a Lipschitz function, then Ω is an extension domain for L^p_k, for all $1<p<\infty$ and $k\in Z^+$. Stein [14] extended this result to include the endpoints $p=1, \infty$ and moreover constructed an extension operator completely independent of k (as well as p). The class of known extension domains was enlarged by Jones [10], who showed that (ε, δ) domains (defined below) are also extension domains for L^p_k, $1\leqq p\leqq\infty$ and $k\in Z^+$. Furthermore, (ε, ∞) domains are extension domains for the Dirichlet space of functions (modulo constants) with gradients in $L^n(\mathbf{R}^n)$ and for BMO [9]. This class of domains is relatively sharp: if $\Omega\subset\mathbf{R}^2$ is a bounded finitely connected extension domain for L^2_1, then Ω is an (ε, ∞) domain.

Ω is an (ε, δ) domain if there are constants $\varepsilon\in(0, \infty)$ and $\delta\in(0, \infty)$ such that