Criteria for absolute convergence of multiple Fourier series

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1. Introduction

The classical theorem of Bernstein can be generalized to the form (Bochner [1, p. 376] and Wainger [3, Theorem 15, p. 78]):

(i) *If a function* $f(t_1, \ldots, t_n)$ *is periodic in each variable and belongs to* Lip(α) *with* $\alpha > n/2$ *then its Fourier series converges absolutely (if* α *is an integer then* Lip(α) *means* C^α; *otherwise it means functions whose partial derivatives of order* $[\alpha]$ *are in* Lip$(\alpha-[\alpha])$ *in the ordinary sense).*

(ii) *There exists a periodic function* $f(t_1, \ldots, t_n) \in$ Lip$(n/2)$ *whose Fourier series does not converge absolutely.*

In this paper we present certain estimates for the absolute sums of Fourier series (Theorem 1 below) and derive criteria for the absolute convergence (Corollary) which are more precise than (i). In analogy with (ii) we show that our criteria, and thus also the underlying estimates cannot be very much improved (Theorem 2).

2. Main results

Let $m=(m_1, m_2, \ldots, m_n)$, where $m_1, \ldots, m_n$ are integers, $t=(t_1, \ldots, t_n) \in R^n$ and $e^{imt}=e^{i(m_1t_1+\cdots+m_nt_n)}$. Let $\sum_m f_m e^{imt}$ be the Fourier series of a function $f(t)$, integrable on $T^n=\{t\colon 0 \leqq t_k \leqq 2\pi;\ k=1, \ldots, n\}$ and 2π-periodic in each variable. We denote $\|f\|_A=\sum_m |f_m|$ and $\|f\|_2=\|f\|_{L_2(T^n)}$. If $\partial^q f/\partial t_k^q \in L_2(T^n)$ for some $q=0, 1, 2, \ldots$ (as usual, $\partial^0 f/\partial t_k^0 \equiv f$) then we put

$$\omega_{j,k}^{(q)}(f, y) = \left\| \frac{\partial^q f}{\partial t_k^q}(t_1, \ldots, t_j+y, \ldots, t_n) - \frac{\partial^q f}{\partial t_k^q}(t_1, \ldots, t_n) \right\|_2.$$