On determinacy notions for the two dimensional moment problem

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One of the most important notions in the theory of the classical moment problem is that of determinacy. In the multidimensional case this concept is very far from being understood. In addition to the "usual" determinacy, Fuglede [5] proposed two other notions which he called strong determinacy and ultradeterminacy. The aim of the present paper is to contribute to a better understanding of these concepts. We restrict ourselves to the two-dimensional case. In Section 1 we discuss three examples. Among others they answer a question of Fuglede ([5], p. 62) by showing that determinacy does not imply strong determinacy and strong determinacy does not imply ultradeterminacy. In Section 2 we introduce another determinacy concept called strict determinacy and we give sufficient conditions for this.

First we explain some terminology and some notation, cf. [5] for more details. Let $d\in\mathbf{N}$. A **positive semi-definite** d**-sequence** is a real d-sequence $\alpha=(\alpha_k)_{k\in\mathbf{N}_0^d}$ such that $\sum_{r,s=1}^{m}\alpha_{k_r+k_s}\gamma_r\overline{\gamma_s}\geqq 0$ for arbitrary $k_1,\ldots,k_m\in\mathbf{N}_0^d$, $m\in\mathbf{N}$ and $\gamma_1,\ldots,\gamma_m\in\mathbf{C}$. Let $\mathscr{H}_\alpha$ denote the canonical Hilbert space associated with a positive semi-definite d-sequence α, see e.g. [5], Section 4. Scalar product and norm of $\mathscr{H}_\alpha$ are denoted by $\langle .,.\rangle_\alpha$ and $\|.\|_\alpha$, respectively. For notational simplicity we consider polynomials of the polynomial algebra $\mathbf{C}(x_1,\ldots,x_d)$ directly as elements of $\mathscr{H}_\alpha$. (Strictly speaking, we have to take the corresponding equivalence classes.) The multiplication operator by a polynomial p on $\mathbf{C}(x_1,\ldots,x_d)$ is denoted by M_p.

Let $M(\mathbf{R}^d)$ be the set of all positive Borel measures μ on $\mathbf{R}^d$ which have finite moments of all order, i.e. $\int|x^k|\,d\mu(x)<\infty$ for all $k\in\mathbf{N}_0^d$. Here we set $x^k:=x_1^{k_1}\ldots x_d^{k_d}$ for $x=(x_1,\ldots,x_d)$ and $k=(k_1,\ldots,k_d)\in\mathbf{N}_0^d$, where $x_j^0:=1$. A real d-sequence $\alpha=(\alpha_k)_{k\in\mathbf{N}_0^d}$ is called a **moment** d**-sequence** if there exists a measure $\mu\in M(\mathbf{R}^d)$ which has the moments α_k, i.e. $\alpha_k=\int x^k\,d\mu$ for $k\in\mathbf{N}_0^d$. In this case we say that μ is a representing measure for α. For $\mu_1,\mu_2\in M(\mathbf{R}^d)$, we write $\mu_1\sim\mu_2$ if μ_1 and μ_2 have the same moments, i.e. $\int x^k\,d\mu_1=\int x^k\,d\mu_2$ for all $k\in\mathbf{N}_0^d$. For a measure μ on $\mathbf{R}^d$ and $j\in\{1,\ldots,d\}$, the projection measure μ_{x_j} is defined as the image of μ under the mapping $(x_1,\ldots,x_d)\to x_j$.