Fredholm pseudo-differential operators on weighted Sobolev spaces

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1. Introduction

Let $m \in (-\infty, \infty)$. Define S^m by

$$S^m = \{\sigma \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n): |D_x^\beta D_\xi^\alpha \sigma(x, \xi)| \leqq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}\}.$$

If $\sigma \in S^m$, then we define the pseudo-differential operator T_σ with symbol σ on $\mathcal{S}$ (the Schwartz space) by

$$(T_\sigma f)(x) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} \sigma(x, \xi) \hat{f}(\xi)\, d\xi, \quad f \in \mathcal{S}.$$

It can be shown that T_σ can be extended to a linear operator from the space $\mathcal{S}'$ of tempered distributions into $\mathcal{S}'$.

Suppose that $\sigma \in S^0$. Then it is well known that T_σ is a bounded linear operator from $L^p(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$ for $1<p<\infty$. An immediate consequence of this result is that every T_σ with $\sigma \in S^m$ is a bounded linear operator from $L^p_{s+m}(\mathbf{R}^n)$ into $L^p_s(\mathbf{R}^n)$ for $1<p<\infty$ and $-\infty<s<\infty$. Here $L^p_s(\mathbf{R}^n)$ stands for the Sobolev space of order s. See Calderón [2] or Stein [19, Chapter 5]. Prompted by the L^p-boundedness result, it is obviously of interest to characterize the nonnegative functions w on $\mathbf{R}^n$ for which every T_σ with $\sigma \in S^0$ is a bounded linear operator on $L^p(\mathbf{R}^n, w\,dx)$ for $1<p<\infty$.

Let $1<p<\infty$. A nonnegative function w is said to be in $A_p(\mathbf{R}^n)$ if $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty$$

where the supremum is taken over all cubes Q in $\mathbf{R}^n$. See Coifman and Fefferman [5] and Muckenhoupt [17] for basic properties of functions in $A_p(\mathbf{R}^n)$. Miller has recently shown in [16] that a necessary and sufficient condition for every T_σ