Behavior of maximal functios in $\mathbf{R}^n$ for large n

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1. Introduction

Let M denote the standard maximal function representing the supremum of averages taken over balls in $\mathbf{R}^n$, that is,

$$M(f)(x) = M^{(n)}f(x) = \sup_{0<r} c_n \frac{1}{r^n} \int_{|y|\leqq r} |f(x-y)|\, dy,$$

where c_n^{-1} is the volume of the unit ball. It has recently been proved (see [2]), that the L^p bounds for $M, p>1$, can be taken to be independent of n. Namely one has

Theorem A. *We have*

(1.1) $$\|M^{(n)}(f)\|_p \leqq A_p \|f\|_p, \quad 1 < p \leqq \infty,$$

with a constant A_p *independent of* n.

What is noteworthy here is that any of the usual covering arguments lead only to a weak-type (1,1) bound which grows exponentially in n, and thus by interpolation one obtains by this method (1.1) with A_p replaced by a bound which increases exponentially in n.

Thus the following further questions now present themselves:

(1) Does $M^{(n)}$ have a weak-type (1, 1) bound independent of n?

(2) What can be said when the usual balls are replaced by dilates of more general sets?

We give here some partial answers to these questions:

(a) First, let B be any bounded, open, convex, and symmetric set in $\mathbf{R}^n$, and let $B^r = \{x | r^{-1}x \in B\}$, $r>0$. Define $M = M_B$ by

$$M_B(f)(x) = \sup_{r>0} (m(B^r))^{-1} \int_{B^r} |f(x-y)|\, dy.$$