Homeomorphisms of the line which preserve BMO

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Suppose $u\colon \mathbf{R}\to\mathbf{R}$ is an increasing homeomorphism. For a function f we then define $(Uf)(x)=f(u^{-1}(x))$. The purpose of this paper is to classify those u for which the operator U is bounded on BMO, the space of functions of bounded mean oscillation. Our theorem answers a question of Coifman and Meyer, and was announced in [2]. Though some time has passed since then, the result still seems to be of some interest as it has been used in several papers. The corresponding problem for $\mathbf{R}^n$, $n\geqq 2$, was solved by Reimann [5], who showed that U is bounded on BMO if and only if u is quasiconformal. The translation of this statement to $\mathbf{R}$, i.e. that u is quasiregular, is false.

In order to understand our result, we must first recall the definition of the Muckenhoupt class A_∞. A positive Baire measure μ is said to be in A_∞ if there are constants C, $\delta>0$ such that whenever I is an interval and $E\subset I$,

$$\frac{\mu(E)}{\mu(I)}\leqq C\left(\frac{|E|}{|I|}\right)^{\delta},$$

where $|\cdot|$ denotes Lebesgue measure. Thus, every $\mu\in A_\infty$ satisfies $d\mu(x)=\omega(x)\,dx$. For such a positive weight function ω, we put $\omega(E)=\int_E \omega\,dx$. We will need to use the fact that whenever u is an increasing homeomorphism, the measure $u'\in A_\infty$ if and only if the same is true for u^{-1}. See [1] for a proof.

Theorem. *The following conditions are equivalent:*

a) $U\varphi\in BMO$ *whenever* $\varphi\in BMO$ *is lower semicontinuous.*
b) U *is a bounded mapping from* BMO *to* BMO.
c) U *is a bijection from* BMO *to* BMO.
d) $u'\in A_\infty$.

The proof of the theorem relies upon two useful facts. The first is the theorem of John and Nirenberg [4]. Let I denote a generic (bounded) interval, and let