On the comparison principle in the calculus of variations

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1. Introduction

A well-known phenomenon in classic Potential Theory can be regarded as a prototype for the variational problem to be studied in this paper. Recall that the **harmonic functions** in a domain $G \subset R^n$, $n \geqq 2$, are precisely the **free extremals** for Dirichlet's integral $\int |\nabla u|^2 dm$.

The basic fact is that the following two conditions are equivalent for a function u with continuous first partial derivatives $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ in G:

1° For every non-negative η in $C_0^\infty(G)$
$\int |\nabla u|^2 dm \leqq \int |\nabla (u-\eta)|^2 dm$
where the integrals are taken over the set $\operatorname{spt} \eta = \overline{\{x | \eta(x) \neq 0\}}$.

2° Given any domain D with compact closure $\bar{D}$ in G and any function h that is harmonic in D and continuous in $\bar{D}$, the boundary inequality $h|\partial D \geqq u|\partial D$ implies that $h \geqq u$ in D.

These conditions express that u is **subharmonic** in G. (Condition 1° is usually formulated as the familiar inequality $\int \nabla u \cdot \nabla \eta \, dm \leqq 0$ for all $\eta \geqq 0$ in $C_0^\infty(G)$.)

The object of our paper is the proper analogue to the above situation for variational integrals of the form

$$I(u, D) = \int_D F(x, \nabla u(x))\, dx, \quad D \subset G. \tag{1.1}$$

Here the integrand is assumed to satisfy certain natural conditions about measurability, strict convexity, and growth: $F(x, w) \approx |w|^p$, $1 < p < \infty$.

If $u \in C(G) \cap W^1_{p,\mathrm{loc}}(G)$ satisfies the inequality

$$I(u, \operatorname{spt} \eta) \leqq I(u-\eta, \operatorname{spt} \eta) \tag{1.2}$$

for every non-negative η in $C_0^\infty(G)$, then u necessarily obeys the comparison principle with respect to the free extremals for the integral (1.1). The corresponding