Some remarks on Banach spaces in which martingale difference sequences are unconditional

J. Bourgain

Introduction

This note deals with Banach spaces X which have so-called UMD-property, which means that X-valued martingale difference sequences are unconditional in $L_X^p(1<p<\infty)$. These spaces were recently studied in [2], [3], [4] and we refer the reader to them for details not presented here. Let us recall following fact (see [2]).

Theorem. *For a Banach space X, following conditions are equivalent:*
(i) *X has* UMD,
(ii) *There exists a symmetric biconvex function ζ on $X\times X$ satisfying $\zeta(0,0)>0$ and $\zeta(x,y)\leqq\|x+y\|$ if $\|x\|\leqq 1\leqq\|y\|$.*

If X has UMD, then the same holds for subspaces and quotients of X, X^* and for the spaces L_X^p $(1<p<\infty)$. It is shown in [1] that if $1<p<\infty$ and $L_X^p(0,1)$ has an unconditional basis, then X is UMD. Conversely, it is not difficult to see that if X is a UMD-space possessing an unconditional basis, then the spaces $L_X^p(0,1)$ $(1<p<\infty)$ have unconditional basis.

In [3], it is proved that if X is UMD, then certain singular integrals such as the Hilbert transform are bounded operators on L_X^p $(1<p<\infty)$. Our first purpose will be to show the converse, i.e. Hilbert transfoım boundedness implies UMD.

From [1], we know that UMD implies super-reflexivity. Another, more direct argument will be given in the remarks below. In [7], an example is described of a superreflexive space failing UMD. We will show that superreflexivity does not imply UMD also for lattices, a question left open by [7].

At this point, the class UMD seems rather small, in the sense that the only basis examples we know about are spaces appearing in classical analysis.