K-functionals and moduli of continuity in weighted polynomial approximation

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1. Introduction

The concept of K-functional was introduced and studied by Peetre ([10], [11]). If A_0 and A_1 are normed linear spaces, both contained in a topological vector space A, then the K-functional is defined by

$$K(A_0, A_1, f, t) = \inf\{\|f_0\|_{A_0}+t\|f_1\|_{A_1}\colon f=f_0+f_1, f_0\in A_0, f_1\in A_1\} \tag{1}$$

Let $A_0=C_{2\pi}=$space of all 2π-periodic continuous functions with $\|f\|_C=\max_{x\in[-\pi,\pi]}|f(x)|$ and $A_1=C'_{2\pi}=$space of all 2π-periodic functions vanishing at 0 and with derivatives in $C_{2\pi}$ with $\|f\|_{C'}=\max_{x\in[-\pi,\pi]}|f'(x)|$. Peetre obtained ([12]) an explicit expression for the K-functional in this case as follows.

$$K(C_{2\pi}, C'_{2\pi}, f, t) = \tfrac{1}{2}\,\omega^*(f, 2t) \tag{2}$$

where ω^* is the least concave majorant of the modulus of continuity of f. It is well-known that this majorant is equivalent to (of the same order of magnitude as) the modulus of continuity of the function. (See, for example, [8]). Such an equivalence can also be obtained between the modulus of continuity of r^{th} order and the K-functional between $C_{2\pi}$ and the space of all 2π-periodic r-times differentiable functions vanishing at 0 along with the first $(r-1)$ derivatives. ([13], [2]). The relation between the K-functionals and the trigonometric approximation is now evident.

For weighted approximation on the whole real line by polynomials, we have obtained in [7], the direct and converse theorems entirely in terms of the K-functionals. Earlier, Freud had introduced a first order modulus of continuity in $L^p(\mathbf{R})$ and proved that this is equivalent to a suitable K-functional ([5]). He considers weights of the form $w_Q(x)=\exp(-Q(x))$ where $Q(x)$ is an even, convex, $C^2(0,\infty)$ function with