On the spectral synthesis problem for points in the dual of a nilpotent Lie group

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1. Introduction

Let A be a *-semi-simple Banach algebra with involution *. One of the main problems concerning the structure of A is the determination of the space $\mathcal{I}$ of the twosided closed ideals of A. Let $\mathrm{Prim}_*(A)$ be the space of the kernels of the topologically irreducible unitary representations of A equipped with the Jacobson topology. For I in $\mathcal{I}$, let $h(I)=\{J\in \mathrm{Prim}_*(A)|J\supset I\}$; $(h(I)$ is a closed subset of $\mathrm{Prim}_*(A))$ and define for the closed subset C of $\mathrm{Prim}_*(A)$ the subset $\mathcal{I}_C$ of $\mathcal{I}$ by $\mathcal{I}_C=\{I\in\mathcal{I}|h(I)=C\}$. The closed subset C of $\mathrm{Prim}_*(A)$ is called a *set of spectral synthesis* if $\mathcal{I}_C$ consists only of one point, namely the ideal $\ker C=\bigcap_{\mathcal{J}\in C}J$. The spectral synthesis problem has been most intensively studied for the algebra $A=L^1(G)$, where G is an abelian, locally compact group G. The first result was the famous theorem of N. Wiener who showed that the empty set is a set of synthesis in $\mathrm{Prim}_* L^1(\mathbf{R})$. The latest deep results are those of I. Domar. (see for instance [4]).

Almost nothing is known for the algebra $L^1(G)$ is G is not abelian. If G is a connected, simply connected nilpotent Lie group, the dual space $\hat{G}$ is well known and thus also the space $\mathrm{Prim}_*(L^1(G))$.

Let $\mathfrak{g}$ be the Lie algebra of G and Ad^* the coadjoint action of G on $\mathfrak{g}^*$. By Kirillow's theorem and Brown's proof of the Kirillow conjecture ([7], [2]) $\hat{G}$ is homeomorphic with the orbit space $\mathfrak{g}^*/_{\mathrm{Ad}^*(G)}$ and [1] tells us that $\mathrm{Prim}_*(L^1(G))\cong\mathfrak{g}^*/_{\mathrm{Ad}^*(G)}$. Thus we may indentify the closed subsets C of $\mathrm{Prim}_*(L^1(G))$ with the closed G-invariant subsets of $\mathfrak{g}^*$. $L^1(G)$ has a remarkable property: For every closed subset C of $\hat{G}$ there exists a twosided ideal $j(C)$ in $L^1(G)$ with the properties:

1) $h(j(A))=A$; 2) $j(A)$ is contained in every closed, twosided ideal I of $L^1(G)$ with $h(I)\subset A$ ([11]).

If G is a group of step 1 and of step 2 every point in $\hat{G}$ is a set of spectral synthesis [9]. In this paper we show that in general a point is not a set of synthesis