Basis properties of Hardy spaces

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1. Introduction

Set $I=[0, 1]$ and let $(\chi_n)_1^\infty$ denote the Haar orthogonal system. If $f\in L^1(I)$ we write $Gf(t)=\int_0^t f(u)\,du$, $t\in I$. Let m be an integer, $m\geqq 0$, and let $(f_n^{(m)})_{n=-m}^\infty$ denote the system of functions which is obtained when we apply the Gram-Schmidt orthonormalization procedure to the sequence of functions $1, t, t^2, \ldots, t^{m+1}, G^{m+1}\chi_2, G^{m+1}\chi_3, G^{m+1}\chi_4, \ldots$ on I. We use here the usual scalar product in $L^2(I)$. The systems $(f_n^{(m)})$ are called spline systems and in particular $(f_n^{(0)})$ is called the Franklin system. These systems are complete in $L^2(I)$ and have been studied by e.g. Z. Ciesielski and J. Domsta [6]. We shall write f_n instead of $f_n^{(m)}$ and set $f_n(t)=0$ for $t\in\mathbf{R}\setminus I$.

For $n\geqq 2$ we have $n=2^j+l$ where $j\geqq 0$, $1\leqq l\leqq 2^j$, and set $t_n=(l-1/2)2^{-j}$. Then $D^m f_n$ is absolutely continuous on I and it is known that

$$|D^k f_n(t)| \leqq M n^{k+1/2} r^{n|t-t_n|}, \quad 0\leqq k\leqq m+1, \quad n\geqq 2, \quad t\in I, \tag{1}$$

where M and r are constants depeding only on m and $0<r<1$ (see [6], p. 316).

Assume that ψ belongs to the Schwartz class of functions $S(\mathbf{R})$ and that $\int_{\mathbf{R}}\psi(x)\,dx\neq 0$. Set $\psi_t(x)=t^{-1}\psi(x/t)$, $t>0$, $x\in\mathbf{R}$, and for $f\in S'(\mathbf{R})$

$$f^*(x) = \sup_{t>0} |f*\psi_t(x)|, \quad x\in\mathbf{R}.$$

The Hardy space $H^p(\mathbf{R})$, $0<p<\infty$, is then defined to be the space of all f such that $\|f\|_{H^p}=\|f^*\|_p<\infty$, where $\|g\|_p$ is defined as $\left(\int |g(x)|^p\,dx\right)^{1/p}$.

For $\alpha>0$ we set $N=[\alpha]$, where [] denotes the integral part, and $\delta=\alpha-N$. If α is not an integer set

$$\dot{\Lambda}_\alpha = \left\{\varphi\in C^N(\mathbf{R});\ \sup_{h\neq 0} \|\Delta_h D^N\varphi\|_\infty/|h|^\delta < \infty\right\}$$

(here $\Delta_h F(x)=F(x+h)-F(x)$) and if α is an integer set

$$\dot{\Lambda}_\alpha = \left\{\varphi\in C^{N-1}(\mathbf{R});\ \sup_{h\neq 0} \|\Delta_h^2 D^{N-1}\varphi\|_\infty/|h| < \infty\right\}.$$