Riemann's zeta-function and the divisor problem

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1. Introduction and statement of the results

This work was finished when the author was visiting the Institute Mittag-Leffler in spring 1981. The support from the Royal Swedish Academy of Science is gratefully acknowledged.

It was found by Atkinson (see [1], [2]) that the mean value problem for $\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$ has "more than superficial affinities" with the classical Dirichlet divisor problem.

Let

$$I(T) = \int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt,$$

$$D(x) = \sum_{n\leq x} d(n),$$

where $d(n)$ denotes the number of positive divisors of n. In [1] Atkinson reproved Ingham's result, viz.

$$I(T) = T\log(T/2\pi)+(2\gamma-1)T+E(T) \tag{1.1}$$

with

$$E(T) \ll T^{1/2+\varepsilon}, \tag{1.2}$$

via the formula

$$I(T) = 2\pi D(T/2\pi)+O(T^{1/2+\varepsilon}).$$

(Atkinson's proof is also presented in Titchmarsh [11], pp. 120—122.) An application of Dirichlet's formula

$$D(x) = x\log x+(2\gamma-1)x+\Delta(x) \tag{1.3}$$

with the elementary estimate $\Delta(x)\ll x^{1/2}$ yields now (1.1) — (1.2). In (1.1) and (1.3) γ denotes Euler's constant.